MISCELLANIES;

CHIEFLY

ADDRESSES,

ACADEMICAL AND HISTORICAL.

BY

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ON LIBERAL TEACHING OF MATHEMATICS.

In so far as Mathematics belongs to universal education, the object of the teacher is *liberal* and not *technical;* hence to make his pupils rapid or even accurate calculators, is of very secondary importance to that of giving them a true insight into the nature of calculation itself. When this has been obtained, they can of themselves improve by practice. Nevertheless, the end of liberal culture is never won, by those who *so* value the effects of the science on the mind, as to undervalue the science itself. No effects at all will follow, unless the pupil aims to master the science, as in itself a worthy reward of exertion. Hence, it is not sufficient to talk or hear *about* the solutions of problems and the performing of operations; but the mind must be called out into activity, and must learn to solve and calculate. In short, here, as in all education, our great aim is, not to communicate mere *results,* but to impart a *power*: and the difference of these two things is so great and important, as to deserve to be dwelt upon a little.

He who learns the dry events and dates of history, learns a *result:* so does he who learns the weights of minerals,—the temperature at which various substances melt,—or the relation between the three sides of a right-angled triangle. All of these results may become *powers,* in the hands of one who knows how to use them; but unless the reasoning faculties are duly cultivated, and the relation of one truth to another is well understood, the knowledge of these things is liable to be a mere useless burden on the memory. In a more and more advanced state of knowledge, every truth becomes an instrument towards the discovery of new truths beyond; and as far as may be, a pupil ought to learn what use to make of the truths communicated. In teaching the mechanical arts, it would avail little to exhibit a chisel or an axe or a turning-lathe, or to expatiate on the qualities in which the perfection of these tools consists, unless the hearer saw them actually used, and learned to handle them; after which, they become to him new *powers*. Till he has seen them used,—and that, slowly enough to allow his mind to follow the process,—they can at best excite barren wonder. To speak generally, the civilized man is eminently distinguished by the variety, power and complexity of his tools. The savage often uses an oystershell or a bit of bone or a stone-hatchet, with a persevering skill that astonishes us; but with these he aims only at immediate results, and not to generate new powers. Yet a beautifully carved mace (cut perhaps with a shell and a nail), being an end in itself and serving no end beyond itself which a rough club might not serve, is far less valuable than a chisel and a plane, which may perform an infinity of other work. So too in more abstract knowledge we uniformly find the successive advances of science marked by the increasing value set on elaborate tools. Such a tool, for instance, is the dictionary of a foreign language; which, as a result in itself, is of no interest except to a professional linguist; yet as a power for opening knowledge beyond, may be of immense value. Such a tool again is every mathematical table,—say, a Multiplication Table, or a Table of Logarithms. In itself it cannot have much interest; but to him who knows how to handle this tool and discerns its various applicability, it ceases to be dry and barren.

In all learning therefore, the pupil needs to see (if possible) what use the teacher makes of his knowledge when he has got it; for that is learning to use the tool. Herein lies the great difficulty of making lectures on History instructive. On this account the *manner* of teaching is sometimes of as much importance as the thing taught: and in Proper Science it is peculiarly to be desired that the mode of proceeding from step to step should, as far as possible, be such as can be used in an infinity of cases widely diverse from that before us. Such was LA PLACE'S advice to mathematical teachers:— "Adhere," says he, "*to general methods*, and you will see your pupils advance."—The elementary and easy parts of every mathematical subject generally admit of being treated by special methods, which fail when applied to higher problems. Such special methods LA PLACE disapproves, because they are not available to the learner as a new power. If on the contrary the easy first problems are treated by *general* methods, which will equally avail afterwards for harder questions, the pupil is hereby armed beforehand. When he approaches an after problem, perhaps he can even solve it of himself, and this is a delightful discovery to him. *General* methods of reasoning are on this account familiarly called "*powerful* methods" by mathematicians; and it is emphatically characteristic of modern mathematics to be comparatively incurious of all special results,—such as, the properties of this or that elegant curve,—except when they take their place in the great arsenal of the science, as instruments that subserve further investigation.

There was a time,—some reminiscences of which still linger among us,—when it was a feat much valued, to find the ratio of the circumference to the diameter of a circle, accurate to a large number of decimal places; and he who had calculated it to 20 decimals seemed to have conferred a greater boon on the science than he who had reached 14 decimals. Now however, the only thing cared for is,—not, to have *actually* achieved such or such a definite result, but—to be *able* to achieve by moderate time and effort, whatever accuracy any real problem may require.

But to treat the questions of the *shop* as the only *real* problems of Arithmetic, appears to me quite illiberal. Nay, it seems more instructive and better, to say as little as possible about shillings and pence, about feet and inches. In every problem, let us take but a single unit of each kind, as a pound, a mile an hour. Let us deal with fractional parts, first as in Vulgar Fractions, and teach all their properties; next, as in Decimals. Whoever once understands these well and on principle, will easily adapt them to any special requirements. Again, let the pupil be familiarized with many forms of serial progression, all in pure numbers; first the natural numbers 1, 2, 3, 4...; then the odd and the even separately; then by addition of the natural numbers you have 1, 3, 6, 10, 15... and by addition of the odd numbers you get the squares 1, 4, 9, 16,... It then becomes an interesting problem, to prove that the last, *however far carried*, will be squares. Further, the idea of two numbers *increasing together* becomes thus familiar, and may be in many ways illustrated; as the foundation for the theory of Variable quantities. In Geometry this may be taught from the very beginning: thus, the Area and the Circumference of a circle, vary with the Radius, and each of them is a "function" of the radius. But in my view, all the operations of elementary arithmetic should be taught by methods as general as those of algebra, and with algebraic signs, but not *letters:* so also, all the simple problems of arithmetic should be solved exactly in the same way as an algebraist obtains his equations, *i.e.* by common sense, and without any technical rule. So soon as problems arise which demand some analysis or *inverse* proceeding, the algebraic x or unknown quantity should be introduced, and the pupil will find no need of rules. In this way arithmetic leads towards and merges in alegbra. Letters a b c belong to the second or even third stage, not to the first.

It is natural for Art to go before Science; and practical problems, eliciting Art, will be in each case our best introduction to Science. But every process of art is best remembered, and grave errors are most certainly avoided, when the *reasons* for the process are learned *soon after* the process itself. A perfect agreement as to the very best mode seems to be here unattainable, and perhaps to different minds different methods are best adapted. I mean, for instance, that it cannot be absolutely settled, whether a pupil ought *first* to be taught how to extract the square root of a number, and *afterwards* learn the reasons, or whether the reasons should go first. Provided, however, that the rules and their reasons be never very far separated, pupils of average intelligence will learn successfully in either way. Nor is it by any means essential, that general theorems be demonstrated in the form which an abstract mathematician would choose. For beginners it is sufficient, and is generally better, to spare to the utmost the formalities of proof, and be satisfied with such a mode of statement as shall carry conviction to their minds that the truth is absolute and necessary. One mystery indeed of mathematics is the singular difficulty of *proving* various truths which certainly ought to be proved, yet which the mind discerns as true at first statement. With beginners I would rather enunciate these as principles to be assumed, than trouble them with the proof. The great point is, to lead the mind *to discern for itself*, and not to rest on *memory* for the rules which are to guide the art and practice. But if it be not trained to such discernment, not only is the difficulty of remembering any complicated operation great, but if any error be made, it will be as easily an enormous as a venial one. And this is universally a danger in that system. A boy who performs arithmetical operations without knowing why, will often use the Rule of Three *Direct* where it ought to be the Rule of Three *Inverse*. Altogether, this makes the whole system of current "Rules" a very dangerous one: for so long as the pupil needs a rule, he needs also to be told which of several rules is the right.

When the moderns universally hold so comprehensive views in Mathematics, I have never understood with what consistency so many Englishmen still uphold the works of

EUCLID as practically valuable to learners, in the present stage of science. This admiration indeed is still warmer in appearance than in fact; for the very greatest liberties have been taken with EUCLID's works. Of his 12 books we never read the 7th, 8th, 9th, 10th—and of the 12th book few read more than the two first propositions. The definitions of the 5th book have been remodelled in almost all the modern treatises from ROBERT SIMSON downwards, who in other parts also has introduced considerable alterations. In more recent times, LEGENDRE in France, PLAYFAIR in Scotland, ELRINGTON in Ireland, have largely deviated from EUCLID'S methods: it is difficult therefore to avoid suspecting that the continued adherence to SIMSON'S Euclid in England is connected with the general attachment of our schools to antiquity. Be this as it may, it seems impossible to deny that the whole plan of EUCLID'S Geometry is the very reverse of that which LA PLACE'S apophthegm would recommend. Instead of imparting general methods, he limits us to specialities. He gives us narrow results, which are not powers. He does not lead the learner to aspire at any thing higher than has been set before him. Nay, his definitions are generally so narrow in their conception, as to shut up the mind in immature and inaccurate notions. He even sometimes gives a succession of different definitions of the same term, and none satisfactory. Thus he first defines *similarity* in the case of similar arcs of circles. A totally different definition of it is given in regard to similar triangles; and a third, in the case of similar rectilinear figures. After all, none of these definitions will apply to similar curves, nor does the learner gain a hint that such curves may exist. Nay, the circle is treated as the only imaginable curve, and the technical words introduced in connection with it are unduly limited or imperfectly explained. Thus in regard to the *contact* of a straight line and a circle, the sole definition of Contact used by him, is, that the line "shall meet and not cut;" which is both insufficient and unnecessary. For the tangent line may sometimes cut a curve; namely at a point of flexure as in the middle of an S; and when this happens, the contact is far more intimate than is possible with the circle. Indeed the whole subject of curvature, extremely important as it is, is evaded, not treated, by EUCLID;—naturally: for one who ignores all curves but the circle, will never understand even circular curvature: and according to EUCLID'S great admirer, ROBERT SIMSON, the characteristic excellence of his 5th book is that it *evades* the difficulties of incommensurate ratio. This might be a merit, if all that we cared for was the establishing of certain theorems convenient in mensuration: but it is surely a great defect, if it hides from the learner, (as in fact it does carefully hide,) the nature of the difficulty to be encountered.

These remarks are not intended to depreciate EUCLID'S merits *in his own day*. He was, among the Greeks, a useful and honoured geometer, just as HIPPOCRATES was an eminent physician. But if any modern school of medicine were to enact the study of HIPPOCRATES as the first step for medical pupils, no doubt it would become a duty with many, so to write concerning the defects of HIPPOCRATES'S knowledge as to seem to undervalue him. The same is the case with EUCLID. His notions, compared to ours, were necessarily very limited, and therefore he could not take a commanding view of his subject: and if his *logic* were ever so perfect, he would still exercise a very cramping effect on a mind which fell beneath his influence. It is generally very unfair to give to young pupils any problems of difficulty to be solved by EUCLID'S methods; for it is ten to one that the teacher himself solves them only by some higher method. No *power* is imparted in the few books of Euclid put into the pupil's hand, though an active intellect may develop for itself ways of analysis out of the synthesis put before it.

I cannot but feel, that his whole treatment of the Straight Line, the Plane, Parallels, and Proportion, is extremely uninstructive. Curvature, of course, he did not understand. He seems more anxious to establish his theorems, than to enlighten the student as to the essence of what he is talking about. On all these subjects, almost a new treatise is needed as a commentary on Euclid, to undeceive or at least to enlighten the learner, who is plunged into difficulties gratuitously.

Again, it has pleased EUCLID to limit his demonstrations by the arbitrary rule, that he will never conceive of any mathematical form which he has not shown how to construct *by rule and compass.* To confine himself to these two instruments was natural, but is needless ; and the effect has been to destroy all compactness of beauty in his treatise, and to give it a most miscellaneous and disorderly aspect. Subjects which ought to be joined are *dis*joined, and the reader does not know *why* he is led away to this and that new subject. EUCLID appears to take his steps like a man floundering in deep snow, who seldom walks straightforward, but plants his foot wherever he guesses he shall find a bottom. When we look back at the path we have traversed, we find, no doubt, that it has brought us to our resting place ; but it has been as winding as a river. This arises out of the feebleness of the method. The ancient road was forced to take shape from the materials of the soil ; the modern one drives straight at its end, unimpeded by any such obstacles. In fact, so conscious was EUCLID of the unshapely character of his treatise, that, in books which we omit, he reduced his results to a more orderly form ; and after treating of ratios in so peculiar a manner in his 5th book, he treated them anew in another book by means of numbers. But for the low state of arithmetic and algebra in that day, he perhaps would never have preferred the form which he has given to Geometrical science. Other points may be here touched. All Mathematical science is difficult to teach *in a class,* because of the disparity in the minds of pupils.—But the difficulty is greatest, when, as generally happens in Euclid, the propositions are so ranged in a single continuous chain that to miss one link makes all fall to the ground. To avoid this, all the propositions of science ought (I think,) as far as possible, to be deduced out of first principles, and not merely one from another. In this way indeed, not only is the accidental loss of a part less fatal, but a deeper appreciation of the science itself is attained, and a far stronger conviction that it is not merely consistent with itself, as some mere hypotheses may be, but is based on irrefragable truth. This tends altogether to sounder habits of thought. To have a firm confidence in the truth of broad principles, is one of the virtues in which English natures are peculiarly apt to be deficient; and mathematical culture fails in one of its great aims, if it does not impart this. Once more, Geometrical Truth ought to be so cultivated as to impart a feeling of Geometrical Beauty. To the uninitiated the very name of this may seem ridiculous, and the satirical poem (or rather its name) The Loves of the Triangles, may be suggested. But it is certain that many great mathematicians have had an enthusiastic feeling of Beauty in their science, and one of them wrote a high flown apostrophe (or I might almost say) *hymn* to the Equiangular Spiral, as the type of Resurrection. What is more, unless the *imagination* is stimulated by the perceptions of Beauty and Symmetry, it is doubly difficult for the *memory* to retain mathematical truth. The teaching therefore should so exhibit the reasonings, as to be not only intelligible, but also elegant.

In Geometry, I think that the general idea of a Limit, as well as of a Ratio, should be explained among the Elements, and that every definition, as well as every process, should be made from the first as general as possible. The idea of *Loci* and of *generating* the surfaces or lines of which we need to speak, as also of Variable Quantities should be made prominent from the very beginning; and the learner should soon be made aware that a Circle is only one out of an infinity of possible curves. But more particularly would I press, that to get a full and intimate conception of all the *Definitions* is of higher value than to remember the miscellaneous *properties* of figures. Some properties indeed are so characteristic, that they are exchangeable with definitions; these particularly need to be known. But there is no worse evil in mathematics than not to know distinctly what it is we are talking about; hence definition is the first great matter. Moreover even without studying the higher mathematics, we may understand many statements made concerning discoveries in them, if only we are acquainted with the definitions, and this is a reason for learning the meaning of words which belong to far higher investigations than we are disposed to pursue. Those who do not read Trigonometry or technical Astronomy will do well to understand such terms as sine, cosine, ellipse, parabola, latitude, poles; and I believe that even those persons who have no taste for the *reasonings* of mathematics find pleasure in getting clearer ideas on such subjects. To become intimate with things themselves is the way to reason soundly concerning them; and geometrical forms are many of them elegant and interesting. To generate them mechanically is the best basis of definition. The idea that to introduce Motion into Geometry, confounds it with Mechanics, is a fundamental mistake. Motion introduces the science of Mechanics, only when we take cognizance of Force, Time, Velocity.

I cannot but think that the Metaphysical part of Geometry should be reserved to the end; and that in the earlier treatment we ought to use, (as freely as we find convenient) any such Assumptions or Postulates as every learner will discern to be rightful. But *they must be prominently set forth as Assumptions*, and never smuggled in as Definitions. This is the very mischievous procedure of our current Euclid, in regard to the Plane and to Proportion. In regard to the Straight Line and Parallels the treatise is more honest; for it propounds *Axioms* concerning them. But as these are axioms in a different sense from most of the Axioms, there is still something to be regretted. That the 12th Axiom concerning Parallels is very ill-chosen, we have a right to assert, since we find it to be abandoned in the best modern treatises, such as LEGENDRE'S. Yet I confess, I have a deeper complaint:

it is, that the *definition* of Parallels is false. For: what are Parallel Circles? Surely not those which, being prolonged ever so far, never meet. Nay, but Circles which are everywhere *equidistant.* This is the only true definition. Parallelism means nothing but Equidistance. So correct the definition, and the postulate needed will not easily be taken amiss. The following will do. "If a straight line has *two* points in it equidistant from another straight line in the same plane, *all* points in it are equidistant from that second line." Then, by corollary, the two lines are Parallel. * * * * * * * * * * * * * * * *