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are $3+\frac{1}{4}, 3+\frac{1}{5}, 3+\frac{1}{6}$, where the denominator we begin with is *the first integer greater than the half of 7*: similarly, that before we come to

$$3+\frac{1}{7}+\frac{1}{15},$$

we have

$$3+\frac{1}{7}+\frac{1}{8},$$

$$3+\frac{1}{7}+\frac{1}{9},$$

$$3+\frac{1}{7}+\frac{1}{10},$$

and so on. When an *even* partial denominator occurs, we take as the partial denominator to begin with, either its half or the first integer greater than its half, according as the partial denominator following is greater or less than that preceding, or, these being equal, according as the next following is less or greater than the next preceding, and so on.

Another improvement, though verbal, is important, viz. in regard to the term *convergent*, the present definition of which seems arbitrary and unreasonable. With great convenience it may be defined as follows:—*A convergent of a fractional number is a fraction which is a closer approximation to the given number than any other fraction with a smaller denominator*; so that Lagrange's problem is simply *to find all the convergents of any fraction.*

On the Relation between two continued Fraction Expansions for Series. *By* THOMAS MUIR, *M.A., F.R.S.E.*

On the Use of Legendre's Scale for Calculating the First Elliptic Integral. *By* Professor F. W. NEWMAN.

Denoting the first elliptic integral by $F(c, \omega)$, and taking x such that $x : \frac{1}{2}\pi = F(c, \omega) : F(c, \frac{1}{2}\pi)$; then, in Lagrange's scale, from ω we deduce successively ω_1, ω_2, ω_3 .... by a given law, with the aid of c_1, c_2, c_3 .... previously determined from c. Then x is the limit to which $\omega, 2^{-1}\omega_1, 2^{-2}\omega_2, 2^{-3}\omega_3$ .... converge. If c is moderately small, the convergence is rapid. But if c^2 is very near to 1, it may be expedient to reverse the direction of the new amplitudes and moduli, viz. to calculate c backwards c', c'', c''', so as to make $c''', c'', c', c, c_1, c_2$, ... a series continued by a single law; and similarly from ω calculate backwards $\omega', \omega'', \omega'''$ ..... Then ω', ω'', ω''' .... are proved to converge to a fixed limit $\omega^{\backprime}$ and $F(c, \omega) : F(b, \frac{1}{2}\pi) =$ Nap log tan $(\frac{1}{4}\pi + \frac{1}{2}\omega^{\backprime}) : \frac{1}{2}\pi$. The function Nap log tan $(\frac{1}{4}\pi + \frac{1}{2}\omega^{\backprime})$ involves but a single element $\omega^{\backprime}$, and was calculated by Legendre. Gudermann has since published a far ampler table. In practice the limit $\omega^{\backprime}$ is quickly reached: often it suffices to make $\omega^{\backprime} = \omega'$, at worst $\omega^{\backprime} = \omega''$. Thus for very large values of c^2 Lagrange's scale practically suffices, presuming that we have at hand tables of $F(c, \frac{1}{2}\pi)$ and $F(b, \frac{1}{2}\pi)$.

But Legendre, who discovered a new scale after completing his principal calculations, regarded his new scale as having much advantage in finding $F(c, \omega)$ at once rapidly and accurately. In it x is the limit of $\omega, 3^{-1}\omega_1, 3^{-2}\omega_2, 3^{-3}\omega_3$ ...., and the convergence, generally excellent in Lagrange's scale, is far more rapid in Legendre's. In Lagrange's scale the relation of ω_1 to ω is $\tan(\omega_1 - \omega) = b \tan \omega$. The relation in Legendre's scale is to the eye as simple, viz. $\tan \frac{1}{2}(\omega_1 - \omega) = A \tan \omega$; but in the constant $A, = \sqrt{(1 - c^2 \sin^2 \beta)}$, the value of β is determined by the equation $F(c, \beta) = \frac{2}{3} F(c, \frac{1}{2}\pi)$. A practical difficulty arose in the very considerable trouble needed to obtain A (or its logarithm) numerically when c was given. Legendre showed how β was obtainable from c: the cubic equation arising can be solved by a mere extraction of the cube-root; but there are also two quadratics involving two extractions of the square-root. Then from β we have to calculate $\sqrt{(1 - c^2 \sin^2 \beta)}$

and find its logarithm before we can proceed to deduce ω_1 from ω. All these operations have to be repeated to find ω_2 from ω_1; nay, we must first find c_1 from c, and that is still more arduous.

But when we assume ρ, $=\frac{1}{2}\pi\frac{F(b, \frac{1}{2}\pi)}{F(c, \frac{1}{2}\pi)}$, as argument, all is greatly simplified. The relation of $c, c_1, c_2, c_3 \ldots$ in Lagrange's scale corresponds with $\rho, 2\rho, 2^2\rho, 2^3\rho, \ldots$, and in Legendre's scale with $\rho, 3\rho, 3^2\rho, 3^3\rho \ldots$, which involve no trouble in calculating. No doubt we need tables (of single entry and easily compiled) to yield c, b when ρ is given, and ρ when c is given. Presuming these, we may treat x and $F(c, \omega)$ as functions of ρ and ω; after which the difficulties of the constant multiplier A vanish, and Legendre's scale becomes practical to us.

Denote $-\log A$, *i. e.* $-\log \sqrt{(1-c^2 \sin^2\beta)}$, for the moment, by $\Phi(\rho)$ (here the common log is intended); then, among the numerous series which express functions of the amplitude ω in terms of x and ρ, the author selects (with λ for Napier's log)

$$-\tfrac{1}{2}\lambda \,.\, \sqrt{(1-c^2 \sin^2\omega)} = \frac{1-\cos 2x}{\sin 2\rho} + \frac{1}{3}\frac{1-\cos 6x}{\sin 6\rho} + \frac{1}{5}\frac{1-\cos 10x}{\sin 10\rho} + \&\text{c.},$$

where $\sin\rho$ is written for $\frac{1}{2}(e^{\rho}-e^{-\rho})$. By hypothesis, $F(c, \beta)=\frac{2}{3}F(c, \frac{1}{2}\pi)$; hence when $\omega=\beta$, $x=\frac{1}{3}\pi$, and we get, writing cosec ρ for the reciprocal of $\sin\rho$, $\frac{1}{3}\Phi(\rho) = M\{\text{cosec } 2\rho + \frac{1}{5}\text{ cosec } 10\rho + \frac{1}{7}\text{ cosec } 14\rho + \frac{1}{11}\text{ cosec } 22\rho + \&\text{c.}\}$, M being the modulus of the common logarithms.

Assuming that we have a table of $\Phi(\rho)$, then given ρ and ω we have the equation $\log\tan\frac{1}{2}(\omega_1-\omega) = \log\tan\omega - \Phi(\rho)$ to find ω_1; $\log\tan\frac{1}{2}(\omega_2-\omega_1) = \log\tan\omega_1 - \Phi(3\rho)$ to find ω_2; $\log\tan\frac{1}{2}(\omega_3-\omega_2) = \log\tan\omega_2 - \Phi(3^2\rho)$ to find ω_3, and so on. The approximation is sufficient when $\Phi(3^n\rho)$ is negligible; and this result is obtained so rapidly, that in the extreme case of $\rho=\frac{1}{2}$, $x=3^{-2}\omega_2$ is correct to ten decimals.

To bring the method to a practical trial, the author has calculated to twelve decimals a skeleton table of $\Phi(\rho)$ for $\rho=0{\cdot}5$, $0{\cdot}6$, $0{\cdot}7$, $0{\cdot}8$, $0{\cdot}9$, and from $\rho=1$ to $\rho=14{\cdot}3$ at intervals of $0{\cdot}1$. The table is given in the paper, and also examples of the method. The process also by which the table was constructed, with the aid of tables of cosec ρ and $e^{-\rho}$, previously calculated by the author, is explained.

General Theorems relating to Closed Curves. By Professor P. G. Tait.

The closed curves contemplated are supposed to have nothing higher than double points. By infinitesimal changes of position of the branches intersecting in it, a triple point is decomposed into three double points, a quadruple point into six, and generally an $x$ple point into $\frac{x(x-1)}{1\,.\,2}$ double points. (1) A closed curve cuts any infinite unknotted line in an even number of points [infinite here implies merely that both ends are outside the closed curve]. (2) The same is true if the line be knotted. (3) If any two closed curves cut one another, there is an even number of points of intersection. (4) In going continuously along a closed curve from a point of intersection to the same point again an even number of intersections is passed. (5) Hence in going round such a closed curve we may go alternately above and below the branches as we meet them. (6) By (3) the same proposition is true of a complex arrangement of any number of separate closed curves superposed in any manner. (7) In passing from the interior of any one cell to that of any other—in any system of superposed closed curves—the number of crossings is always even or always odd, whatever path we take. (8) Hence the cells may be coloured black and white in such a way that from white to white there is always an even number of crossings, and from white to black an odd number. Such closed curves therefore divide the plane as nodal lines do a vibrating plate.

The above are the enunciations of the propositions proved in the paper, which, with the necessary figures &c., will be found printed *in extenso* in the 'Messenger of Mathematics,' vol. vi., January 1877.
