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On Tridiametral Quartan Curves. By F. W. Newman.

Problem. To find the conditions that a Quartan may have 3 Diameters.

That it may have one, the equation must admit the form

$$ay^4+X_2y^2=X_4,$$

where X_n means a function of x of the nth degree.

Let $r^2=x^2+y^2$; then we may write

$$ar^4+(Ax^2+Bx+C)r^2=kx^4+lx^3+mx^2+nx+p.$$

This *form* will not be changed if we change the origin to any point in the axis of y; hence, if there be a second diameter, we may may suppose it to pass through the origin, which we treat as a Pole, making $x=r\cos\psi$, $y=r\sin\psi$.

Then $ar^4+(Ar^2\cos^2\psi+Br\cos\psi+C)r^2=$

$$kr^4\cos^4\psi+lr^3\cos^3\psi+mr^2\cos^2\psi+nr\cos\psi+p=0,$$

which by the routine of trigonometry is expressible as

$$\begin{aligned}&\left\{a-\tfrac{3}{8}k+\tfrac{1}{2}(A-k)\cos 2\psi-\tfrac{1}{8}k\cos 4\psi\right\}r^4\\+&\left\{(B-\tfrac{3}{4}l)\cos\psi-\tfrac{1}{4}l\cos 3\psi\right\}r^3\\+&\left\{(C-\tfrac{1}{2}m)-\tfrac{1}{2}m\cos 2\psi\right\}r^2=nr\cos\psi+p.\end{aligned}$$

This is the equation of every Quartan which has so much as one diameter.

In order that the line expressed by $\psi=\gamma$ may be a new diameter, it is necessary and it suffices that the same equation should result by substituting $\psi=\gamma+\omega$, and $\psi=\gamma-\omega$, where γ is a definite constant, r, ω the variables of the equation. Put $\psi=\gamma\pm\omega$; then in order that $\pm$ may give the same result, the terms concerned must vanish in the coefficients of r^4, r^3. r^2, r separately. It must be observed that the assumption $\gamma=0$ or $\gamma=180^\circ$ is useless; and $\gamma=90^\circ$ leads us to two rectangular diameters, not to three. Hence we must avoid to suppose $\sin\gamma=0$ or $\sin 2\gamma=0$.

Now (1) $nr\sin\gamma\,.\sin\omega=0$, $\therefore n=0$;

(2) $m\sin 2\gamma\sin 2\omega=0$, $\therefore m=0$;

(3) in the coefficient of r^3, we need at once

$$(B-\tfrac{3}{4}l)\sin\gamma\sin\omega=0;\ \tfrac{1}{4}l\sin 3\gamma\,.\sin 3\omega=0.$$

It is useless to suppose $l=0$, $B=0$; for this, joined with $m=0, n=0$, reduces the equation to the Doubly Diametral. Hence our only useful results are

$$\sin 3\gamma=0,\ B=\tfrac{3}{4}l;\ \text{which leave B and } l \text{ finite.}$$

(4) $(A-k)\sin 2\gamma\,.\sin 2\omega=0, k\sin 4\gamma\sin 4\omega=0.$

We cannot make $\sin 4\gamma=0$, since we already require $\sin 3\gamma=0$. Hence nothing remains but $k=0$, $A=0$.

Thus the original equation is reduced to

$$ar^4+(Bx+C)r^2=\tfrac{4}{3}Bx^3+p; \quad \dots\dots\dots \quad (h)$$

and from $\sin 3\gamma=0$ we get *two* new diameters, defined by $\gamma=60^\circ$ and $\gamma=180^\circ$. Thus the problem is solved.

Originally, the assumption $a=0$ would have left our monodiametral curve still a Quartan. But after supposing $A=0$ and $k=0$, we cannot make a also $=0$ without reducing the equation to a Tertian. In fact it is easy to show that the conditions here investigated yield the known Tertian *Trijuga* when we add the assumption $a=0$.

Writing $x=r\cos\psi$, $4x^3=r^3(\cos 3\psi+3\cos\psi)$, we find

$$ar^4+Cr^2=\tfrac{1}{3}Br^3\cos 3\psi+p, \quad \dots\dots\dots \quad (i)$$

which is the most general Polar Equation of Tridiametral Quartans.

Again, solving (h) for r^2, and making $a=1$, since a must be finite,

$$r^2+\tfrac{1}{2}(Bx+C)=\sqrt{\{\tfrac{4}{3}Bx^3+\tfrac{1}{4}(Bx+C)^2+p\}}.$$

Thus the general equation to rect. coords. has the form

$$y^2+x^2+B'x+C'=\sqrt{\{\tfrac{8}{3}B'x^3+(B'x+C')^2+E\}}, \quad \dots\dots \quad (j)$$

which has 3 Parameters.

If, however, $C'=0$ and $E=0$, the Polar equation becomes simply $r=\frac{2}{3}B' \cos 3\psi$, which is a Starry Trijuga, admitting $r=0$.

In general, the equation to rect. coords. falls under the class

$$y^2+X_2=\sqrt{X_3},$$

which is the highest form of those which I call *Quartotertian*.

The Polar equation may be presented in the form $\cos 3\psi=\frac{Ar^4+Br^2+C}{r^3}$.

The curve is evidently in every case finite, and the species must apparently change according as the equation admits the forms $ar^3 \cos 3\psi=(r^2-b^2)(r^2-c^2)$, $ar^3 \cos 3\psi=(r^2+b^2)(r^2-c^2)$, $ar^3 \cos 3\psi=(r^2+b^2)(r^2+c^2)$, or finally $ar^3 \cos 3\psi=(r^2\pm b^2)^2+c^4$. Evidently $\frac{dr}{dx}=0$, when $\sin 3\psi=0$.

If $\psi=120^\circ+\theta$, $\cos 3\psi=\cos 3\theta$. Hence the figure is Equilateral.

On Quartan Curves with 3 or 4 Diameters. *By* F. W. Newman.

This Memoir proposes and solves the Problems, in what case Curves of the Fourth Degree have 3 or 4 diameters.

It briefly analyzes the forms of the Tridiametral Curves, under the heads which rise out of the general equation

$$2ar^3 \cos 3\psi=r^4+2br^2+c=R:$$

1. when $R=r^4$, or $2a \cos 3\psi=r$;
2. when $R=r^4-\beta^2r^2$, or $2ar \cos 3\psi=r^2-\beta^2$;
3. when $R=r^4\beta^2r^2$;
4. when $R=r^4-\gamma^4$;
5. when $R=r^4+\gamma^4$, and generally when R is essentially positive;
6. when $R=(r^2-\beta^2)(r^2-\gamma^2)$, which has 3 remarkable forms;
7. when $R=(r^2+\beta^2)(r^2-\gamma^2)$, which has 2 forms, according as β^2 is $>\gamma^2$ or $<\gamma^2$.

On Monodiametral Quartan Curves. *By* F. W. Newman.

This Memoir is a continuation of the paper laid before the Association last year on Doubly Diametral Quartan Curves, and follows upon a notice now presented on Tridiametrals and Quadridiametrals of the same degree.

Employing X_n to mean an integer function of x, of degree n, it is proposed to digest all the Monodiametral curves into five Groups, twenty-one Classes, as follows:—

I.
1. $y^4+2Ay^2=X_1$, or x an integer function of y^2 [*Quartic* Parabolas].
2. $y^4+X_1y^2=X_1$, or x rational in y^2 (Conic Parabola for asymptotes).

II. (In these, y^2 is rational in x.)
3. $X_2y^2=X_1$ }
4. $X_2y^2=X_2$ } (Horizontal and Vertical asymptotes).
5. $y^2=X_4$ (two equal and opposite Conic Parabolas for asymptotes).
6. $xy^2=X_4$; or, the Semicubical; with *Tertian* asymptote.
7. $X_2y^2=X_3$ (Conic Parabola for asymptote).
8. $X_2y^2=X_4$; Quartohyperbolic.

III.
9. $y^2=\sqrt{X_3}$; Quartotertian of 1st Branch.
10. $y^2\pm A^2=\sqrt{X_3}$; Quartotertian of 2nd Branch.
11. $y^2+X_1=\sqrt{X_1}$ (Epiparabolic asymptote).
12. $y^2+X_1=\sqrt{X_2}$ (Unequal Conic Parabolas for asymptotes). [$y^2=\sqrt{X_2}$ is omitted as Doubly Diametral.]
13. $y^2+X_1=\sqrt{X_3}$; Quartotertian of 3rd Branch (*Epi*parabolic asymptote).

IV.
14. $y^2+X_2=\sqrt{X_1}$ (*Epi*hyperbolic asymptote).
15. $y^2+X_2=\sqrt{X_2}$ (admits two asymptotic hyperbolas, with their rectilinear asymptotes *parallel*, set to set).
16. $y^2+X_2=\sqrt{X_3}$; Quartotertian of 4th Branch. (*Tridiametrals* must be excluded.) Epihyperbolic asymptotes.

V.
17. $y^2 = \sqrt{X_4}$ (Quartohyperbolic Group).
18. $y^2 + A^2 = \sqrt{X_4}$.
19. $y^2 + X_1 = \sqrt{X_4}$ (one Hyperbolic asymptote at most).
20. $y^4 + X_2 y^2 = X_3$ or X_2 or X_1 (perhaps one Parabolic and one Hyperbolic asymptote).
21. $y^4 + X_2 y^2 = X_4$ or $y^2 + X_2 = \sqrt{X'_4}$ (perhaps two Hyperbolic asymptotes, all differently directed).

The mode of analysis used in the most difficult cases is as follows:—

It is assumed that if $\phi\ (v\ x) = 0$, and $f\ (u\ x) = 0$ are known curves, and $y'^2 = v^2 + u^2$, $y''^2 = v^2 - u^2$, the curve $F\ (y,\ x) = 0$ can thence be traced, y'^2 and y''^2 being the two positive roots of y^2, when such are real. Practically it is not difficult to decide on the course of $(y\ x)$, if the constants which enter the two auxiliaries are fixed; but the number of hypotheses concerning the relations of the constants in ϕ to the constants in f are embarrassing.

Thus, to trace (y, x) from the equation $y^2 + X_2 = \pm\sqrt{X_4}$, which is the largest case, we put $X_2 = -y_1^2$, or else $X_2 = +y_0^2$, according to the sign which X_2 may assume within different limits, and $Y^2 = \sqrt{X_4}$. Then either $y^2 = Y^2 - y_0^2$, giving at most only one positive value to y^2; or $y^2 = y_1^2 \pm Y^2$, giving in some cases two positive values to y^2.

This assumes that we know not only y_1 and y_0, which define Conic curves, but also $Y^2 = \sqrt{X_4}$. If X_4 degenerate, $Y^2 = \sqrt{X_1}$ is a Quartic Parabola. $Y^2 = \sqrt{X_2}$ is a *Doubly* Diametral Quartan, which is here assumed to be known; $Y^2 = \sqrt{X_3}$ is the primary Quartotertian (9th Class of Quartans); $Y^2 = \sqrt{X_4}$ is the primary Quartohyperbolic of the 17th class. Thus the 9th class becomes auxiliary to the 10th, 13th, and 16th; and the 17th is auxiliary to all which follow it. The 1st class (Quartic Parabola) is auxiliary to the 11th and 14th.

It is believed that in the 8th class alone there are in strictness as many as 260 species. This makes it impossible to undertake to draw them all, which multiply more and more in the higher classes, as the number of constants increase. Nevertheless many diagrams are laid before the Association, nearly exhausting the forms of the earlier classes. The Semicubical and the Quartotertian are notable as peculiarly novel and most remote from the Doubly Diametral.

Many of the forms might be conjectured beforehand from the Doubly Diametral by merely introducing inequality, as in place of two equal, two unequal ovals. Nevertheless there is much that could never be so conjectured, just as in the Doubly Diametral we could not conjecture the forms of the inferior classes from knowing the superior forms.

On the Circular Transformation of Möbius. *By* Prof. H. J. Stephen Smith, *F.R.S.*

General Physics.

On Sympathy of Pendulums. By Professor P. G. Tait, *F.R.S.E.*

On Relations between the Gaseous, the Liquid, and the Solid States of Matter. By Prof. James Thomson, *LL.D., Queen's College, Belfast.*

The object of this paper is to submit some new theoretical considerations which constitute a further development of one portion of the views offered, at last year's Meeting of the Association, by the author, in his paper entitled "Speculations on the Continuity of the Fluid State of Matter, and on Relations between the Gaseous, the Liquid, and the Solid States." He has now to make reference to the abstract