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VOL. I.

ὅ τι οὐσία πρὸς γένεσιν, ἐπιστήμη πρὸς πίστιν καὶ διάνοια πρὸς εἰκασίαν ἐστι.

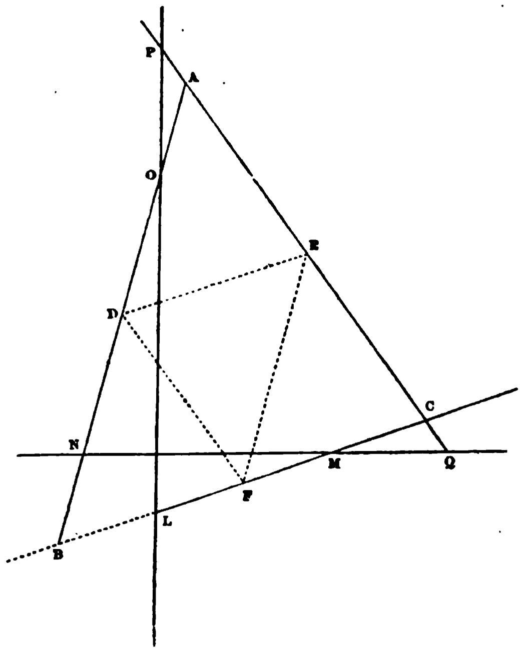
LONDON:

JOHN W. PARKER AND SON, WEST STRAND.

1857.

DGF = two right angles ∴ the $\angle NBS = \angle FGD$; and then the proof is the same, and $DN = DG = DO$, and in like manner $PE = EQ$.

Corollary.—If two lines cut each other at right angles, and any two lines *LM*, *NO* be drawn, cutting off two of the



angles, and *LM*, *NO* be bisected in *F* and *D*, and produced to meet in *B*, then if *DA* be taken equal to *DB*, and *FC* be taken equal to *FB*, and the line *AC* be drawn cutting the two lines in *P* and *Q*, *AP* shall equal *CQ*.

SUSPENSION BRIDGE.

By Professor F. W. Newman.

1. I have not been able to learn that the following investigation has anywhere been worked out. If new, it may have practical interest.

In bridges of great span, the weight of the material is a serious difficulty, and it is well to study the theory which will reduce this difficulty to a minimum.

Given the material, and the tension to which it can be safely subject, we shall avoid all needless weight by everywhere *proportioning the strength to the strain*, and hence (if the material is uniform) proportioning the *weight to the strain.* This principle will here be adopted as characteristic.

Let a cylindrical piece of the material (say iron, or wire-rope), which is l in length, with a base whose area is e^2, have the weight w, and be able safely to bear the longitudinal tension t. Thus its tension on every square unit of a transverse section is $\frac{t}{e^2}$ and the weight of every cubic unit is $\frac{w}{le^2}$. For simplicity let the specimen be taken from the lowest point of the chain, and let *one* chain only support the bridge: so that t is the tension of the chain at its lowest point, where also its transverse area is e^2.

Let a be a linear magnitude, determined by the proportion $\frac{w}{t}=\frac{l}{a}$: then, if $W=$ weight of a variable portion of the chain, whose length is s, and T the tension at the end, the principle that *strength varies as strain*, yields the proportion $dW \propto Tds$, or $\frac{dW}{w}=\frac{T}{t}\cdot\frac{ds}{l}$; which in a simpler form is

$$dW=T.\frac{ds}{a}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

an equation which characterises the present inquiry.

Again; in *every* Catenary, the horizontal tension is constant; so that if x is the horizontal ordinate to s,

$$T\frac{dx}{ds}=t\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

from which two, eliminate T;

$$\therefore\frac{dW}{t}=\frac{ds^2}{dx^2}\cdot\frac{dx}{a}.$$

To get some notion of the possible length of a, we may refer to the extreme case of wire-rope. Mr. Andrew Smith's wire-ropes (ROPE, *Penny Cyclopædia*), when they weigh 34 ounces per fathom, can bear a strain of 8 tons. Put $l=$

2 yards, $w=34$ ounces, $t=8$ tons; $\therefore a=\frac{2\times 8\cdot 20\cdot 112\cdot 16}{34}$ yards, which is nearly 9 miles. But of course, for each material, a will be computed, not at its extreme tension, but at its certainly safe tension.

2. For a *first problem,* suppose the chain to hang by its own weight alone. Let the lowest point be the origin of the horizontal and vertical co-ordinates x and y. Then the vertical tension $T\frac{dy}{ds}$, in passing from s to $(s+ds)$ receives the increment dW only; or

$$d\left(T\frac{dy}{ds}\right)=dW.......................(3).$$

The eqq. (1), (2), (3), contain the full solution of this case. Eliminating T and W, put $y'=\frac{dy}{dx}$ for a moment; $\therefore$ we readily get $\frac{dy'}{1+y'^2}=\frac{dx}{a}$; the second differential eq. to the curve. Hence $y'=\tan\frac{x}{a}$; and integrating anew, $\cos\frac{x}{a}=\varepsilon^{-\frac{y}{a}}$, or $\frac{y}{a}=-\log\cos\frac{x}{a}$; the finite eq. to the curve of the chain. Since $\frac{dy}{dx}=\tan\frac{x}{a}$, $\therefore \frac{x}{a}=$ angle of the tangent's inclination to the horizon, which may be called θ.

When $2x=\pi a$, $y=\infty$. Hence this is a span too large to be crossed without exposing the chain to a tension greater than that assigned by hypothesis: and when x at all approaches this magnitude, y is impracticably high. Yet since y is only "logarithmic infinity," its increase for moderately large values of x will not be embarrassing.

We have also

$$\frac{ds}{dx}=\sec\frac{x}{a},\ \frac{s}{a}=\log\tan\left(\frac{\pi}{4}+\frac{x}{2a}\right)=\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta).$$

Likewise

$$2\sec\frac{x}{a}=\varepsilon^{\frac{s}{a}}+\varepsilon^{-\frac{s}{a}}:\ 2\tan\frac{x}{a}=\varepsilon^{\frac{s}{a}}-\varepsilon^{-\frac{s}{a}};$$

$$T=t.\sec\frac{x}{a}:\ W=t.\tan\frac{x}{a}:$$

which entirely complete all that can be asked in the case.

When $\frac{x}{a}$ is small, $\frac{y}{a} = \frac{1}{2} \cdot \frac{x^2}{a^2} + \frac{1}{12} \cdot \frac{x^4}{a^4} + \frac{1}{45} \cdot \frac{x^6}{a^6}$ nearly;

hence the osculating parabola is $2ay = x^2$.

3. *Next,* let a heavy road hang to the chain, but neglect the vertical bars. If a length l of the roadway with its extreme load weighs w' (which we suppose to be uniform), the vertical tension has an additional increment, $w' \cdot \frac{dx}{l}$; which changes eq. (3) into

$$d\left(T\frac{dy}{ds}\right) = dW + w'\,\frac{dx}{l}: \;\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

or if $\frac{w'}{w} = m$,

$$a \,.\, dy' = (1 + y'^2)\,dx + m dx.$$

Put $1 + m = n$,

$$\therefore \quad y' = \sqrt{n} \,.\, \tan\frac{x\sqrt{n}}{a} = \tan\theta\,;\ \cos\frac{x\sqrt{n}}{a} = \epsilon^{-\frac{y}{a}}\,;$$

the finite eq. of the chain.

The limiting span of the bridge is now

$$2x = \frac{\pi a}{\sqrt{n}}.$$

Also

$$\frac{T^2}{t^2} = 1 + n\tan^2\frac{x\sqrt{n}}{a}:\ \frac{W}{t} = \frac{T}{t}\cdot\frac{dy}{ds} - \frac{w'}{t}\cdot\frac{x}{l} = \frac{dy}{dx} - m\,\frac{x}{a}.$$

To compute s, put

$$u = \tan\frac{x\sqrt{n}}{a} \therefore \frac{2\sqrt{n}\,ds}{a} = \sqrt{(1 + nu^2)}\,\frac{2\,du}{1 + u^2}$$

$$= \sqrt{(u^{-2} + n)}\,.\,\frac{2u\,du}{1 + u^2}.$$

Again, put

$$u^{-2} + n = v^{-2}, \therefore \frac{2\sqrt{n}\,ds}{a} = \frac{1}{v}\,d\log\frac{1 - mv^2}{1 - nv^2};$$

or

$$\frac{2s\sqrt{n}}{a} = \sqrt{n}\log\frac{1+v\sqrt{n}}{1-v\sqrt{n}} - \sqrt{m}\log\frac{1+v\sqrt{m}}{1-v\sqrt{m}};$$

where $v^{-2} = n + \cot^2\left(\frac{x\sqrt{n}}{a}\right)$.

Since $v\sqrt{n} = \sin\theta$, let $v\sqrt{m} = \sin\psi$, or $\tan\psi = \sqrt{m}\sin\frac{x\sqrt{n}}{a}$: or $\sin\theta : \sin\psi :: \sqrt{n} : \sqrt{m}$. Then

$$\frac{s}{a} = \log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right) - \sqrt{\frac{m}{n}}\log\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi\right).$$

4. But we ought *not* to neglect the vertical rods, which are practically essential, and which theoretically introduce no new element. Abandoning then eq. (4), or treating it as a first approximation only, let us seek to add to it the strain caused by the weight of the rods.

Imagine the rods to be joined, so as to make a thin vertical parapet connecting the road with the chain. Its thickness at bottom must be uniform, as is the weight of the road; but it must get thicker as it rises higher: moreover, the tension on every point is to be the same as in the chain.

Let τ = thickness at bottom; then τl = area of the horizontal section of the parapet, which supports the weight w' of the road; and $\frac{w'}{\tau l}$ is the strain on every square unit at the bottom of the parapet. But $\frac{t}{e^2}$ is the strain on every square unit of the chain; hence $\frac{w'}{\tau l} = \frac{t}{e^2}$.

Now $w' = mw$, and $\frac{w}{l} = \frac{t}{a}$ $\therefore$ $a\tau = me^2$, which determines τ.

Farther: if at the horizontal distance x, and at any height u, the thickness of the parapet be z, the increment of its vertical weight is proportionate to $z\,du$, which again (by hypothesis) is proportionate to dz; since the parapet must get thicker to bear the increase of strain. Thus $z\,du = c\,dz$.

Integrating, and correcting so that $z = \tau$, when $u = 0$, we get $\frac{u}{c} = \log\frac{z}{\tau}$.

To find c, observe that $\int_0 z\,du = c\,(z-\tau)$: moreover, since the parapet, like the chain, weighs $\frac{w}{e^2 l}$ in every cubic unit, the total weight straining the parapet *at* the level u is $\frac{w'}{l}dx + \frac{wdx}{e^2 l}\int_0 zdu$; the former term being from the road, the latter from the parapet itself *beneath* the level u. This weight is proportional to the area strained, *i. e.* to zdx. Put then C, another constant, such that $Czdx = \frac{w'}{l}dx + \frac{wdx}{e^2 l}$. $c\,(z-\tau)$; or $Cz = \frac{mt}{a} + \frac{tc}{e^2 a}(z-\tau)$; in which z may vary independently. Hence the eq. splits into two; viz. $m = \frac{c\tau}{e^2}$, $C = \frac{tc}{e^2 a}$. The former gives $c=a$; the latter, $C = \frac{t}{e^2}$. This is as it should be; since C = ratio of weight straining, to area strained.

Hence $\frac{u}{a} = \log\frac{z}{\tau}$; which determines z, the thickness of the parapet, at any height u. Where it reaches the chain, $\frac{y}{a} = \log\frac{z'}{\tau}$.

5. Returning to the chain itself, observe that the increment of the vertical strain, in passing from s to $s+ds$, is $dW + Cz'dx$:

$$\therefore d\left(T\frac{dy}{ds}\right) = dW + \frac{t}{e^2}.\,\tau\varepsilon^{\frac{y}{a}}\,dx\ldots\ldots\ldots\ldots\ldots(5.)$$

which is now to be combined with (1) and (2). Eliminate W and T;

$$\therefore a\frac{d^2y}{dx^2} = \frac{ds^2}{dx^2} + m\varepsilon^{\frac{y}{a}}:$$

the *second* differential eq. of the curve.

To integrate, let $\frac{ds^2}{dx^2} = v$, $\therefore \frac{1}{2}a.\frac{dv}{dy} = v + m\varepsilon^{\frac{y}{a}}$. When $m=0$, the integral is $v = c\varepsilon^{\frac{2y}{a}}$. Put then $v = X\varepsilon^{\frac{2y}{a}}$, and you get $dX = 2m\epsilon^{-\frac{y}{a}}\frac{dy}{a}$, which (corrected so that when $y=0$, $\frac{ds}{dx}=1$, $v=1$, $X=1$), gives as the *first* differential eq.

$$1+\frac{dy^2}{dx^2}=(1+2m)\,\epsilon^{\frac{2y}{a}}-2m\epsilon^{\frac{y}{a}}.$$

For a second integration, observe that if $m=0$, $c + \frac{x}{a}$ $= \cos^{-1}(\epsilon^{-\frac{y}{a}})$. Assume, therefore, $\epsilon^{-\frac{y}{a}}=\cos V$, and eliminate y. Then by processes of routine you get $\frac{dx}{a}$

$$=\frac{dV}{\sqrt{(1+m\sec^2.\frac{1}{2}V)}}=\frac{\cos\frac{1}{2}V\,.\,dV}{\sqrt{(\cos^2.\frac{1}{2}V+m\;)}}=\frac{2d\sin\frac{1}{2}V}{\sqrt{(n-\sin^2\frac{1}{2}V)}}:$$

whence, correcting aright, $\sqrt{n}\sin\frac{x}{2a}=\sin\frac{V}{2}$

or $1-\epsilon^{-\frac{y}{a}}=n\left(1-\cos\frac{x}{a}\right)$: finite eq. to the curve......(6).

When $\frac{x}{a}$ is small, its versed sine is much smaller. If for a moment we call it p, we have $\frac{y}{a}=-\log(1-np)$

$$=np+\tfrac{1}{2}n^2p^2+\tfrac{1}{3}n^3p^3+\text{etc.}$$

But as n is considerable, the convergence of this is doubtful.

So also, since $2n$ is >1, the development of $\frac{y}{a}$ in series of multiple arcs has unsatisfactory convergence; viz. if $2n = \sec^2\nu$,

$$\therefore\ \frac{y}{a}=\log\cos^2\nu-\log\left(\cos^2\nu-\sin^2\frac{x}{2a}\right)$$

$$=2\log(2\cos\nu)-2\left\{\cos 2\nu\cos\frac{x}{a}-\tfrac{1}{2}\cos 4\nu\cos\frac{2x}{a}\right.$$

$$\left.+\tfrac{1}{3}\cos 6\nu\cos\frac{3x}{a}-\text{etc.}\right\}$$

Nevertheless, this gives the *area* of our curve (or nearly, of the parapet) with rather better convergence, viz.

$$\int_0^x y\,dx = 2ax \log(2\cos\nu)$$

$$-2a^2\left\{\cos 2\nu \sin\frac{x}{a} - \frac{\cos 4\nu}{2^2}\sin\frac{2x}{a} + \frac{\cos 6\nu}{3^2}\sin\frac{3x}{a} - \text{etc.}\right\}$$

6. Put θ as before; or $\tan\theta = \dfrac{dy}{dx}$.

Then

$$\frac{y}{a} = \log\frac{1+\cos 2\nu}{\cos\frac{x}{a} + \cos 2\nu}\ ,\quad \tan\theta = \frac{\sin\frac{x}{a}}{\cos\frac{x}{a} + \cos 2\nu};$$

where also the denominator $= 2\cos\left(\frac{x}{2a} - \nu\right)\cos\left(\frac{x}{2a} + \nu\right)$, for convenience of logarithms.

Again, from the first differential eq.

$$\sec^2\theta = (1+2m)\,\varepsilon^{\frac{2y}{a}} - 2m\varepsilon^{\frac{y}{a}}.$$

Moreover, we know T from the eq.

$$\frac{T}{t} = \frac{ds}{dx} = \sec\theta.$$

To find W, eq. (5) gives

$$W = T\frac{dy}{ds} - mt\int_0^x \varepsilon^{\frac{y}{a}}\frac{dx}{a},$$

or

$$\frac{W}{t} = \tan\theta - m\int\frac{1+\cos 2\nu}{\cos\frac{x}{a} + \cos 2\nu}\cdot\frac{dx}{a}:$$

To integrate, put $\tan\dfrac{x}{2a} = U$, which yields

$$\frac{1}{\tan\nu}\cdot\log\frac{1+U\tan\nu}{1-U\tan\nu}.$$

Also $\dfrac{m}{\tan\nu} = \cot 2\nu$; hence

$$\frac{W}{t} = \tan\theta + \cot 2\nu\cdot\log\frac{\cos\left(\frac{x}{2a} - \nu\right)}{\cos\left(\frac{x}{2a} + \nu\right)}.$$

To find the solid mass in the parapet, which is $\iiint dz\,du\,dx$, or $\iint z\,du\,dx$, or $\int_0 a(z'-\tau)\,dx$, or $\int_0 a\tau\varepsilon^{\frac{y}{a}}\,dx - a\tau x$; observe that the integral remaining is obtainable by eq. (5) in the form $\frac{ae^2}{t}\left\{T\frac{dy}{ds}-W\right\}$. Farther, observe that every cubic unit weighs $\frac{w}{le^2}$, and remember the relations $wa=tl$, $a\tau=e^2$, $mw=w'$, and you find the *weight* of the parapet to be

$$\left(T\frac{dy}{ds}-W\right)-w'\,.\,\frac{x}{l}\;:$$

which indeed appears by a more direct process, namely that which elicited eqq. (4) and (5).

Thus, weight of parapet

$$=\cot 2\nu\cdot\log.\frac{\cos\left(\frac{x}{2a}-\nu\right)}{\cos\left(\frac{x}{2a}+\nu\right)}-w'.\frac{x}{l}.$$

7. Lastly, to find s, the length of the chain, we may have recourse to functions of θ.

We have $\tan\theta=\dfrac{\sin\frac{x}{a}}{\cos\frac{x}{a}-h}$, if $h=-\cos 2\nu=\frac{m}{m+1}$. Put $H^2=1-2h\cos\frac{x}{a}+h^2$; $\therefore\ \sec\theta=H\div\left(\cos\frac{x}{a}-h\right)$; $\sin\theta =H^{-1}.\sin\frac{x}{a}$, $\cos\theta=H^{-1}\left(\cos\frac{x}{a}-h\right)$; $\sqrt{(1-h^2\sin^2\theta)}$ (which may be called $\Delta\,(h,\theta)$ as in elliptic integrals) $=H^{-1}\left(1-h\cos\frac{x}{a}\right)$. Hence we have $\Delta\,(h,\theta)-h\cos\theta =H$; after which we can solve for functions of x, and get

$$\sin\frac{x}{a}=\sin\theta\,\{\Delta(h,\theta)-h\cos\theta\}\,;$$

$$\cos\frac{x}{a}-h=\cos\theta\,\{\Delta\,(h,\theta)-h\cos\theta\}\,;$$

$$1-h\cos\frac{x}{a}=\Delta\,(h,\theta)\,\{\Delta(h,\theta)-h\cos\theta\}.$$

Again, since $\sec^2\theta\, d\theta$

$$=\left(1-h\cos\frac{x}{a}\right)\frac{dx}{a}\div\left(\cos\frac{x}{a}-h\right)^2, \text{ and } \frac{ds}{dx}=\sec\theta,$$

we get

$$\frac{ds}{a}=\sec\theta\,.\,\frac{dx}{a}=\sec\theta\,.\,\frac{\Delta(h,\theta)-h\cos\theta}{\Delta(h,\theta)}\,.\,d\theta,$$

and

$$\frac{s}{a}=\int_0\frac{d\theta}{\cos\theta}-h\int_0\frac{d\theta}{\Delta(h,\theta)}=F(1,\theta)-hF(h,\theta)$$

or

$$=\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta\right)-hF(h,\theta);$$

where F is the elliptic integral of the first species.

If $m=0$, $h=0$, and we regain the formula of the first problem.

8. When x is so long as to reach a landing, m vanishes. Upon this the curve of the chain may assume advantageously the form of the first problem, if it need to be continued any significant length beyond: but the new curve will have a different origin. Let x, y, θ have the values x_0, y_0, θ_0 at the landing, and the chain be continued beyond. Put $\tan\theta_0$

$$=\frac{\sin\dfrac{x_0}{a}}{\cos\dfrac{x_0}{a}-\left(\dfrac{m}{m+1}\right)}$$ from the original curve, $=\tan\dfrac{x'}{a}$ in the new curve, and determine x' from x_0. Also $\dfrac{y'}{a}=-\log.\cos\dfrac{x'}{a'}$ whence y' is known. It is easy to see that x', y' are greater than x_0, y_0; hence the origin of the new curve is thrust back and depressed by the distances $(x'-x)(y'-y)$ which are easy to calculate.