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V. "On Determinants, better called Eliminants." By Professor FRANCIS NEWMAN, M.A. Communicated by Dr. BOOTH, F.R.S. &c. Received March 6, 1857.

(Abstract.)

1. This paper aimed at recommending the introduction into elementary treatises of the doctrine of Determinants; which, following Professor Boole, it called *Eliminants*. It exemplified the great aid to the *memory* which the notation affords. It undertook to show, that if only so much of new notation be used, as is needed in elementary applications, the subject becomes full as easy as the second part of algebra. The method of proceeding recommended may be understood by the following concise statement.

If n linear eqq. are given, connecting n unknown quantities; and every eq. is represented by $A_r x + B_r x_2 + C_r x_3 + \cdots + N_r x_n = P_r$ (where r is 1, 2, 3 ... n in the several eqq.), then, solving for any one of the unknowns, we of course obtain a result of the form $mx=a$. Very simple considerations then show, that m and a will be *integer functions of the coefficients*: namely, it is easy to prove, that *if* this is true for one number n, it must needs be true also for the number $(n+1)$; and consequently is generally true. Next, the same analysis exhibits, that $m=0$, is the result obtained, when $P\, P_2\, P_3 \ldots P_n$ all vanish: moreover, that if the system presented for solution be the $(n-1)$ eqq.

$$\left.\begin{array}{l} Bv + B_2 v_2 + B_3 v_3 + \ldots + B_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Nv + N_2 v_2 + N_3 v_3 + \ldots + N_n = 0 \end{array}\right\}$$

and the solutions are denoted by

$$m'v = a_1;\ m'v_2 = a_2;\ \ldots m'v_{n-1} = a_{n-1};$$

we get the relations

$$\left.\begin{array}{l} m = Aa_1 + A_2 a_2 + A_3 a_3 + \ldots + A_{n-1} a_{n-1} + A_n m'; \\ a = Pa_1 + P_2 a_2 + P_3 a_3 + \ldots + P_{n-1} a_{n-1} + P_n m'; \end{array}\right\}$$

out of which flow all the rules for the *genesis* of Eliminants, and the application of them to solve linear eqq. of any degree.

In adapting the theory to the proof of elementary propositions, as, in forming the *Product* of two Eliminants, the paper urged the utility of the principle, that every Eliminant is *a linear function of any one of its columns*, and also, *of any one of its rows*;—which principle may often be so applied as to show by inspection, *à priori*, that certain constituents are excluded from this and that function, and thus enable us to obtain its value by assuming arbitrary values for such constituents. It deprecated (at least for elementary uses) the notations used by Mr. Spottiswoode* and others, not only as involving needless novelty to learners, but because no page can be broad enough to afford to write

$(1, 2)(1, 1)'+(2, 2)(1, 2)'+(3, 2)(1, 3)'$ instead of $BX+bY+\beta Z$,

and because accents, so related, are hard to see in a full page, and the general aspect of every element is so like that of every other element, that the fatigue of reading soon becomes confusing and intolerable.

2. But the main topic of the paper was to advocate the use of Eliminants in Geometry of three dimensions, especially in every systematic treatise on Surfaces of the Second Degree. Various illustrations and results were given, which the writer believed to be new; on which account, some of them may be briefly noticed here.

Problem. "To find the length of a perpendicular ρ, dropt from a given point $(a\,b\,c)$, on to a given plane $lx+my+nz+p=0$; when the axes are oblique, and the cosines of the angles $(xy)(xz)(yz)$ are *given*; viz. $=D, E, F$."

Result. Take G and H to represent the eliminants

$$G=\begin{vmatrix} 1 & D & E \\ D & 1 & F \\ E & F & 1 \end{vmatrix} \quad \text{and } H=\begin{vmatrix} 1 & D & E-l \\ D & 1 & F-m \\ E & F & 1-n \\ l & m & n & 0 \end{vmatrix};$$

then ρ is known from the eq.

$$\rho\sqrt{H}=(la+mb+nc+p)\sqrt{G}.$$

When ρ is given, this eq. determines the relations between $l\,m\,n\,p$, which are the test, that the plane may touch a sphere given in position.

* It may be right to state, that Mr. Newman opened the paper by a grateful and honourable recognition of Mr. Spottiswoode's labours.

Problem. To analyse the forms assumed by the locus of the general eq.

$$Ax^2+By^2+Cz^2+2A_2x+2B_2y+2C_2z$$
$$+2Dxy+2Exz+2Fyz+G=0 \text{ (axes oblique).}$$

Result. Let $V=\begin{vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{vmatrix}$ and $W=\begin{vmatrix} A & D & E & A_2 \\ D & B & F & B_2 \\ E & F & C & C_2 \\ A_2 & B_2 & C_2 & G \end{vmatrix}$; then in the common treatises (only without this notation) it is shown that when V is finite, the surface (if real) has a centre. It is here added, that when W is negative, the curvature is everywhere towards the same side of the tangent plane; when W vanishes, the tangent plane coincides with the surface in one straight line; but when W is positive, the surface is *cut* by the tangent plane in two intersecting straight lines, and the curvature bends partly towards one side of the tangent plane, partly towards the other.

Hence it appears that we have different sorts of surfaces, by combining $V=0$ or $V=$finite, with $W=0$ or $W=$positive, or $W=$negative.

The locus is *imaginary*, if W is >0, A and B finite, $CG-C_2^2>0$,

and $C\begin{vmatrix} A & E & A_2 \\ E & C & C_2 \\ A_2 & C_2 & G \end{vmatrix}>0.$

The locus is *degenerate*, if of ABC one at least (as C) be finite,

and if $V=0$, $\begin{vmatrix} A & E & A_2 \\ E & C & C_2 \\ A_2 & C_2 & G \end{vmatrix}=0$, $\begin{vmatrix} B & F & B_2 \\ F & C & C_2 \\ B_2 & C_2 & G \end{vmatrix}=0$: or if A B C all vanish, and if at the same time $D=0$, and $E:F:C_2=2A_2:2B_2:G$.

Problem. To investigate the nature of the plane intersections of the surface.

Result. If the cutting plane be $lx+my+nz+p=0$, the section is a hyperbola, parabola or ellipse, according as $\begin{vmatrix} A & D & E & l \\ D & B & F & m \\ E & F & C & n \\ l & m & n & o \end{vmatrix}$ is positive, zero, or negative.

The intersection *degenerates*, if $\begin{vmatrix} A & D & E & A_2 & l \\ D & B & F & B_2 & m \\ E & F & C & C_2 & n \\ A_2 & B_2 & C_2 & G & p \\ l & m & n & p & o \end{vmatrix} = 0.$

In a non-centric surface, where $V=0$, we readily find that the former of these eliminants has the same sign as (D^2-AB); and consequently, that non-centric surfaces cannot have sections of opposite species. It also appears, that to determine in a non-centric surface the parabolic sections, we must take $l\,m\,n$ such as to verify one of the three eqq.

$$\begin{vmatrix} A & D & E \\ D & B & F \\ l & m & n \end{vmatrix} = 0, \quad \begin{vmatrix} A & D & E \\ l & m & n \\ E & F & C \end{vmatrix} = 0, \quad \begin{vmatrix} l & m & n \\ D & B & F \\ E & F & C \end{vmatrix} = 0.$$

Problem. To determine the circular sections, when they exist.

Result. Take the larger question, of ascertaining when two surfaces of the second degree intersect in a plane curve. Denote the coefficients of the second surface by accents. Put $\alpha = A\rho - A'$; $\beta = B\rho - B'$; $\gamma = C\rho - C'$; &c. and determine ρ by the eq.

$$\begin{vmatrix} \alpha & \delta & \epsilon \\ \delta & \beta & \phi \\ \epsilon & \phi & \gamma \end{vmatrix} = 0;$$

which involves ρ in the third degree.

Then $l\,m\,n$ will be determined (when the surds are real) by the proportion

$$l : m : n = \sqrt{(\epsilon^2 - \alpha\gamma)} + \epsilon : \sqrt{(\phi^2 - \beta\gamma)} + \phi : \gamma.$$

To apply this to the problem of circular sections, it is only necessary to suppose the second surface to be a sphere.

The surface becomes one of Revolution, if (with oblique axes) either system of three eqq. is fulfilled:

$$\begin{cases} (1) \quad \alpha\beta = \delta^2, \quad \alpha\gamma = \epsilon^2, \quad \beta\gamma = \phi^2, \\ (2) \quad \alpha\phi = \delta\epsilon, \quad \beta\epsilon = \phi\delta, \quad \gamma\delta = \epsilon\phi. \end{cases}$$

If out of each triplet we eliminate ρ^2 and ρ, (for it seems easiest to treat these as independent variables,) the result is two eqq. (expressible by eliminants), which are the two general conditions for a surface of revolution.

Problem. To find the system of rectangular conjugates. This of course is cardinal, and is treated everywhere: but is made far easier by Eliminants, as follows. Let us inquire after *that diameter, common to two given concentric surfaces, which shall have its conjugate planes the same for both.*

Take the centre for the origin, and $x=mz$, $y=nz$ for the common diameter sought. Then the central planes conjugate to it in the two surfaces are

$$\left.\begin{array}{l}(\mathrm{A}m+\mathrm{D}n+\mathrm{E})x+(\mathrm{D}m+\mathrm{B}n+\mathrm{F})y+(\mathrm{E}m+\mathrm{F}n+\mathrm{C})z=0\\ (\mathrm{A}'m+\mathrm{D}'n+\mathrm{E}')x+(\mathrm{D}'m+\mathrm{B}'n+\mathrm{F}')y+(\mathrm{E}'m+\mathrm{F}'n+\mathrm{C}')z=0.\end{array}\right\}$$

To identify these two planes, let

$$\frac{\mathrm{A}m+\mathrm{D}n+\mathrm{E}}{\mathrm{A}'m+\mathrm{D}'n+\mathrm{E}'}=\frac{\mathrm{D}m+\mathrm{B}n+\mathrm{F}}{\mathrm{D}'m+\mathrm{B}'n+\mathrm{F}'}=\frac{\mathrm{E}m+\mathrm{F}n+\mathrm{C}}{\mathrm{E}'m+\mathrm{F}'n+\mathrm{C}'}=\frac{1}{\rho},$$

or

$$\alpha m+\delta n+\epsilon=\delta m+\beta n+\phi=\epsilon m+\phi n+\gamma=0.$$

Eliminate m, n, and you find that ρ is to be determined by the very same eq. as in the preceding; and since its eq. is of the third degree, it has always one real value.

Next, let the second surface be a sphere, and you find *at least one* diameter of the first surface *perpendicular* to its conjugate plane. Make this diameter the axis of x, and take for the axes of y and z the two *principal* diameters of the section in the conjugate plane. Then $\mathrm{D}=0$, $\mathrm{E}=0$, $\mathrm{F}=0$; so that the general eq. is reduced to $\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2+\mathrm{G}=0$. Moreover, the system of axes is now rectangular: hence the axis of y, and that of z, equally with that of x, are each perpendicular to its conjugate plane, and the eq. for ρ must have three real roots, corresponding to these three axes.

We might similarly investigate "the conditions of contact for two concentric surfaces;" which, when one of them is a sphere, gives the cubic whose roots are a^2, b^2, c^2, principal axes of an Ellipsoid.

Problem. To discuss the results of *Tangential Co-ordinates.* [This expression is employed as by Dr. James Booth in an original tract on the subject.]

Put $\quad \mathrm{P}=\mathrm{A}x+\mathrm{D}y+\mathrm{E}z+\mathrm{A}_2 \quad \Big| \quad \mathrm{R}=\mathrm{E}x+\mathrm{F}y+\mathrm{C}z+\mathrm{C}_2$

$\quad\quad \mathrm{Q}=\mathrm{D}x+\mathrm{B}y+\mathrm{F}z+\mathrm{B}_2 \quad \Big| \quad \mathrm{S}=\mathrm{A}_2x+\mathrm{B}_2y+\mathrm{C}_2z+\mathrm{G}$

Then $\mathrm{P}x+\mathrm{Q}y+\mathrm{R}z+\mathrm{S}=0$ is the eq. to the surface, and $\mathrm{P}x'+\mathrm{Q}y'+\mathrm{R}z'+\mathrm{S}=0$ is the eq. to the tangent plane at (xyz). Hence if

$x'y'z'$ are the three tangential co-ordinates (or intercepts cut from the co-ordinate axes by the tangent plane) we have $Px'+S=0$, $Qy'+S=0$, $Rz'+S=0$. Let $\xi\ \eta\ z$ be the *reciprocals* of $x'\ y'\ z'$. Then $P+\xi S=0$, $Q+\eta S=0$, $R+\zeta S=0$; and the eq. to the surface becomes $\xi x+\eta y+\zeta z-1=0$. Restore for PQR their equivalents; then eliminating $xyzS$ you get

$$\begin{vmatrix} A & D & E & A_2 & \xi \\ D & B & F & B_2 & \eta \\ E & F & C & C_2 & \zeta \\ A_2 & B_2 & C_2 & G-1 & \\ \xi & \eta & \zeta-1 & 0 & \end{vmatrix} =0;$$

general eq. to the surface, with axes oblique.

If the last eq. (developed) be represented by

$$a\xi^2+b\eta^2+c\zeta^2+2a_2\xi+2b_2\eta+2c_2\zeta+2d\xi\eta+2e\xi\zeta+2f\eta\zeta+g=0,$$

it is not difficult to obtain a system of eqq. in which $abc...\xi\eta\zeta$ play the same part, as just before did $ABC...xyz$. Whence again we have

$$\begin{vmatrix} a & d & e & a_2 & x \\ d & b & f & b_2 & y \\ e & f & c & c_2 & z \\ a_2 & b_2 & c_2 & g-1 & \\ x & y & z-1 & 0 & \end{vmatrix} =0;$$

which is the *original* eq. of the surface under the form of an Eliminant.

The most arduous problems (as Dr. James Booth has shown) are often facilitated by these co-ordinates; but without Eliminants, the eqq. cannot be treated generally and simply.

The paper likewise contained the application of Eliminants to tangential co-ordinates in Curves of the Second Degree; and urged that eliminants ought to be introduced into the general treatment of these curves also, if only in order to accustom the learner to their use and gain uniformity of method. Thus, if the general eq. be

$$Ax^2+By^2+C+2Ex+2Fy+G=0,$$

then $V=0$ is the test of degeneracy.