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ON THE THIRD ELLIPTIC INTEGRAL.

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I. Following up the splendid discoveries of Jacobi, Legendre first, and since his labours were closed, Dr. Gudermann, have investigated series of an elevated kind for approximating to this integral. But the higher theory seems to have drawn off investigations unduly from what is more elementary; and the principal object of the present paper is to shew that the earlier and simpler methods have by no means been adequately appreciated and developed.

Legendre's notation, with trifling alterations, will be here retained. The moduli $cc_1c_2c_3\ldots$ are those of the common scale *descending*, which he denotes by $cc^\circ c^{\circ\circ}c^{\circ\circ\circ}\ldots$ I propose to employ the notation $\eta\eta^\circ\eta_\circ$ to imply the relations

$$F(c\eta)+F(c\eta^\circ)=F(c,\tfrac{1}{2}\pi),$$
$$F(b\eta)+F(b\eta_\circ)=F(b,\tfrac{1}{2}\pi),$$

in which case η° may be called the *conjugate* amplitude to η, and $\eta_\circ$ the *lower conjugate*. We then have the well-known relations $\cot\eta.\cot\eta^\circ=b$,

$$\sin\eta^\circ=\frac{\cos\eta}{\Delta(c\eta)},\quad \cos\eta^\circ=\frac{b\sin\eta}{\Delta(c\eta)},\quad \Delta(c\eta^\circ)=\frac{b}{\Delta(c\eta)}\ \ldots(1),$$

in which we may change η°, c into $\eta_\circ, b$.

Also if from c,η be formed c_1,η_1 in Lagrange's scale, we get

$$\left.\begin{aligned}&\sqrt{c_1}\sin\eta_1=\sqrt{c}\sin\eta.\sqrt{c}\sin\eta^\circ;\quad \Delta(c\eta)+\Delta(c\eta^\circ)=(1+b)\Delta(c_1\eta_1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \Delta(c\eta)-\Delta(c\eta^\circ)=(1-b)\cos\eta_1\\ &\cos\eta_1=\sin(\eta^\circ-\eta),\ldots\qquad \text{or } \eta_1=\tfrac{1}{2}\pi+\eta-\eta^\circ\end{aligned}\right\}\ldots(2).$$

For the complete integrals $F(c,\tfrac{1}{2}\pi)$, $E(c,\tfrac{1}{2}\pi)$, &c.... we may generally conform to a prevalent method of writing them F_c, E_c, &c. But in the case of F_c it is sometimes necessary to break the analogy (as in the higher theory) by writing C: for when c changes to $cc_1c_2\ldots bb_1b_2\ldots$, to write F_{c_1}, F_{c_2}, &c. is very incommodious.

II. Legendre's results concerning the integral Π_1 may be summed up nearly as follows:

(1) That every Π which has a parameter of the form $\alpha+\beta\sqrt{-1}$ is reducible to *two* Π's with real parameters, and with coefficients of the form $\alpha+\beta\sqrt{-1}$.

(2) If $Q=\dfrac{\tan\omega}{\Delta(c\omega)}=\dfrac{\cos\omega^\circ}{b\cos\omega}$, and $T=(1+p)\left(1+\dfrac{c^2}{p}\right)$,

$$\therefore \int \frac{dQ}{1+TQ^2} = \Pi(p) + \Pi\left(\frac{c^2}{p}\right) - F \text{............} (3),$$

where p, c^2p^{-1} are parameters, and c, ω the other elements.

Let $pq = c^2$, then p and q are called *reciprocal* in this theory.

(3) If $(1+p)(1-r) = b^2$, therefore

$$\int_0 \frac{d(\sin\omega\sin\omega^\circ)}{1+pr(\sin\omega\sin\omega^\circ)^2} = \frac{1+p}{p}.\Pi(p) - \frac{1-r}{r}.\Pi(-r) - \frac{c^2}{pr}.F\ldots(4).$$

The parameters p and $-r$ are called *conjugate.*

(4) Various integrals $P(p, \omega) = \int\phi(p, \omega)\,d\omega$ are known, which may be found in some simpler form for *a special* value of ω, $(\omega = \alpha)$ by means of

$$\frac{dP}{dp} = \int\frac{d\phi(p,\omega)}{dp}\,d\omega = \psi(p,\omega).$$

If the function ψ is known, we get

$$\frac{dP(p,\alpha)}{dp} = \psi(p,\alpha), \text{ and } P(p,\alpha) = \int\psi(p,\alpha)\,dp.$$

Legendre applied this method to Π, and deduced not only the value of Π_c in terms of F and E, but certain *commutative* equations, in which the amplitude exchanges places with a certain function of the parameter.

(5) He applied Lagrange's scale to Π, and by it deduced two series, one for descending, the other for ascending, moduli. But both are too complicated for use, especially the latter.

One more property established by Legendre remains to be named; viz. if ζ, ω, η are *amplitudes* which make $F\zeta = F\omega + F\eta$, and p the parameter common to the three Π's which correspond to the F's, then

$$\sqrt{T}.\{\Pi\omega + \Pi\eta - \Pi\zeta\} = \tan^{-1}.\frac{\sqrt{T}.p\sin\omega\sin\eta\sin\zeta}{1+p(1-\cos\omega\cos\eta\cos\zeta)}\ldots\ldots(5).$$

Consequently if $\zeta = \frac{1}{2}\pi$, or $\eta = \omega^\circ$, we get

$$\sqrt{T}\{\Pi\omega + \Pi\omega^\circ - \Pi_c\} = \tan^{-1}.\left\{\frac{p}{1+p}.\sqrt{T}.\sin\omega\sin\omega^\circ\right\}\ldots(6),$$

in which, whenever T is negative, it will be easy to give to the last term the form of a logarithm.

My first business is, to shew that all these integrations of Legendre, when duly simplified, lead to available results.

III. The function $T = (1 + p)(1 + q)$ may be called the *test product*, since, according as it is negative or positive, Π is of the logarithmic or of the circular class. We may occasionally denote it by $T(p)$, and the definition shews that $T(p) = T(q)$: also

$$T(p) . T(-r) = b^4 \text{ .... } (7).$$

It is easy to prove that the reciprocals of conjugate parameters are conjugate, and the conjugates of reciprocals are reciprocal. Thus the reciprocal of the conjugate is the conjugate of the reciprocal.

Also two reciprocals, or two conjugates, are either both circular or both logarithmic.

Legendre assigns two forms for logarithmic parameters, viz. $-c^2 \sin^2 \eta$ and $-\text{cosec}^2 \eta$, both negative; the former ranging from 0 to $-c^2$, the latter from -1 to $-\infty$. Evidently if a parameter has the form $-c^2 \sin^2 \eta$, its reciprocal is $-\text{cosec}^2 \eta$. But logarithmic *conjugates* are either both of the form $-c^2 \sin^2 \eta$ or both of the form $-\text{cosec}^2 \eta$. In fact, since $\Delta(c\eta) . \Delta(c\eta^\circ) = b$, it follows that $-c^2 \sin^2 \eta$ and $-c^2 \sin^2 \eta^\circ$ are logarithmic conjugates. So also, since $\cot \eta . \cot \eta^\circ = b$, therefore $-\text{cosec}^2 \eta$ and $-\text{cosec}^2 \eta^c$ are logarithmic conjugates.

A circular parameter, when positive, may be denoted by $p = \cot^2 \theta$; then its reciprocal is $q = c^2 \tan^2 \theta$, and its conjugate is $-r = -1 + b^2 \sin^2 \theta$. But we may also write $p = \cot^2 \theta$, $q = \cot^2 \theta_\circ$, $r = \Delta^2(b, \theta)$. Of two circular conjugates one is necessarily negative, the other positive. Also, since

$$\Delta(b, \theta) \, \Delta(b, \theta_\circ) = c,$$

the two circular parameters $-\Delta^2(b, \theta)$ and $-\Delta^2(b, \theta_\circ)$ are reciprocal.

In the equations (5), (6) every Π is multiplied by $\sqrt{T}$; and in the farther development of the theory the same phenomenon constantly recurs; insomuch that $\sqrt{T}\Pi$ seems (rather than Π) to be the function which we are concerned with. In this connection it is highly interesting to find, that the curves drawn on surfaces of the second order, which are said to be measured by this integral, are really measured by the compound function $\sqrt{T}.\Pi$. (See Dr. James Booth, *Philos. Trans.* 1852, p. 320, equations (17), (18), &c.) Of course, when T is negative, we must deal with $\sqrt{-T}.\Pi$.

When T is separated from Π, it may be requisite, as above, to write the parameter after T; as $T(p)$ for $(1 + p)(1 + c^2 p^{-1})$, &c. But whenever T is *immediately* followed by Π, (or by P, of which I proceed to speak,) it will be understood that T involves the *same* parameter and modulus as the Π, or as the P. Thus $\sqrt{T}\Pi(p) + \sqrt{T}\Pi(-r)$ will mean

$$\sqrt{T(p)}.\Pi(p) + \sqrt{T(-r)}.\Pi(-r).$$

When p is infinitesimal, $\Pi = F$; also $T = c^2 p^{-1}$; whence

$$\sqrt{T}(F - \Pi) = c\sqrt{p^{-1}}\int_0 p \sin^2\omega . dF = 0.$$

Also when p is infinite, $T = p$, and $\sqrt{T}\Pi$ has no increments while ω is finite; for then $\dfrac{\sqrt{p}}{1 + p\sin^2\omega} = 0$. But while ω is infinitesimal $\sqrt{T}\Pi = \displaystyle\int_0 \frac{\sqrt{p}d\omega}{1 + p\omega^2} = \tan^{-1}(\sqrt{p}.\omega)$, which becomes $\tan^{-1}(\infty)$, as soon as ω rises to a sensible value; hence $\sqrt{T}\Pi = \frac{1}{2}\pi$, when $p = \infty$, whatever the finite value of ω. This reasoning is rather refined; but the conclusion may be equally obtained from the reciprocal equation.

IV. The three integrals F, E, Π have in common the property, that whenever their amplitude $\omega = n.\frac{1}{2}\pi$, the integral $= n$ times the complete integral. It immediately follows, that if for a moment we assume three arcs x, x', x'' such that

$$\frac{F(\omega)}{F_c} = \frac{x}{\frac{1}{2}\pi}, \quad \frac{E(\omega)}{E_c} = \frac{x'}{\frac{1}{2}\pi}, \quad \frac{\Pi(\omega)}{\Pi_c} = \frac{x''}{\frac{1}{2}\pi},$$

each of the three new arcs is equal to ω, as often as ω is a multiple of $\frac{1}{2}\pi$. Hence $(x' - x)$ and $(x'' - x)$, or any functions proportional to them, vanish periodically every time that $\omega = n.\frac{1}{2}\pi$. Such *fluctuating* functions are the appropriate auxiliaries for calculating E and Π, when F is known.

Legendre assumed G as an auxiliary, equivalent to $E - \dfrac{F}{F_c}.E_c$; which is proportional to $x' - x$; and by it he obtained by far the most elegant of the approximations to E; namely, if C now stands for F_c, he found

$$\left.\begin{aligned} CG &= C_1G_1 + C_1c_1 \sin\omega_1 \text{ [Lagrange's scale]} \\ \text{whence}\quad CG &= C_1c_1\sin\omega_1 + C_2c_2\sin\omega_2 + C_3c_3\sin\omega_3 + \&c.... \end{aligned}\right\}...(8).$$

But the properties and uses of G have by no means been fully exhibited, and a digression on that subject, either here or afterwards, is inevitable. If we assume H as a second auxiliary, such that

$$H = E - \left(1 - \frac{E_b}{F_b}\right)F ....(9),$$

we easily get [since by Legendre's equation of complementary moduli, $\frac{1}{2}\pi = F_bE_c + F_cE_b - F_bF_c$,] therefore $$H = G + \frac{\frac{1}{2}\pi F}{F_b F_c};$$

or, in the other notation,

$$H = G + \tfrac{1}{2}\pi \cdot \frac{F}{BC} \ldots\ldots\ldots\ldots(10).$$

Moreover $$CH = C_1 H_1 + C_1 c_1 \sin\omega_1;$$

or $$BH = \tfrac{1}{2}B_1 H_1 + \tfrac{1}{2}B_1 c_1 \sin\omega_1 \ldots\ldots\ldots(11),$$

whence

$$\left.\begin{aligned} BH = B\sin\omega - 2B'(\sin\omega - \sin\omega') - 2^2B''(\sin\omega' - \sin\omega'') \\ - 2^3B'''(\sin\omega'' - \sin\omega''') - \&c. \end{aligned}\right\}\ldots(12),$$

which is by far the most elegant series for calculating E by *ascending* moduli, (*i.e.* when c is near to 1,) and converges with the usual precipitancy of Lagrange's scale.

Farther, if $\sin\eta = \sqrt{-1}\tan\theta$, we easily obtain

$$\left.\begin{aligned} &G(c\eta) = -\sqrt{-1} \cdot H(b,\theta) + \sqrt{-1} \cdot \tan\theta \cdot \Delta(b,\theta), \\ &H(c\eta) = -\sqrt{-1} \cdot G(b,\theta) + \sqrt{-1} \cdot \tan\theta \cdot \Delta(b,\theta), \\ \text{or}\quad &G(c\eta) - \sqrt{-1} \cdot G(b,\theta) = H(c\eta) - \sqrt{-1} \cdot H(b,\theta), \\ \text{also}\quad &G(c\eta) + \sqrt{-1} . G(b,\theta) = \sqrt{-1}\left\{\tan\theta\Delta(b,\theta) - \tfrac{1}{2}\pi . \frac{F(b\theta)}{BC}\right\} \end{aligned}\right\}\ldots(13).$$

Finally, it is worth observing, that while $G(c\omega)$ vanishes, not only when ω is a multiple of $\frac{1}{2}\pi$, but also when c is evanescent; we have, on the other hand, $BH_c = \frac{1}{2}\pi$, for all values of c; also, for all values of ω, we find $BH = \omega$ when $c = 0$; but $H = \sin\omega$, when $c = 1$. We may add that $G(c, n\pi + \omega) = G(c, \omega)$, but $BH(c, n\pi + \omega) = BH(c, \omega) + n\pi$, or $H(c, n\pi + \omega) = H(c, \omega) + 2nH_c$.

V. Returning to the integral Π, we follow out the analogy of this proceeding, by assuming an auxiliary proportional to $(x'' - x)$.

Let P stand for $$\Pi - \frac{F}{F_c}\Pi_c \ldots\ldots\ldots\ldots(14),$$

then the problem of finding Π divides itself into two parts. First, to find the complete integral Π_c: for when this is known, we regard the second term of P to be known. Next, it remains to find the fluctuating portion P, which alone involves three elements; and since it periodically vanishes, we may look on it as a small correction to be applied to the main term; the total value of Π being given by the equation

$$\Pi = \frac{F}{F_c}\Pi_c + P.$$

It is evident that P vanishes with p. But we must first dispose of the case to which this method is essentially inapplicable, viz. that in which Π_c is infinite; namely in which the parameter has the form $p = -\operatorname{cosec}^2\eta$. By the reciprocal equation (3) Legendre reduces this to the indefinite integral $\Pi(-c^2\sin^2\eta)$: nevertheless, it is not amiss to exhibit the equation in a slightly changed form.

When $p = -\operatorname{cosec}^2\eta$, $T = -\cot^2\eta.\Delta^2(c\eta)$, which applies alike to p and q: therefore

$$\sqrt{-T}\{\Pi(p) + \Pi(c^2p^{-1}) - F\}$$
$$= \int_0 \frac{\sqrt{-T}.dQ}{1+TQ^2} = \tfrac{1}{2}\log\frac{\tan\eta\Delta\omega + \tan\omega\Delta\eta}{\pm\{\tan\eta\Delta\omega - \tan\omega\Delta\eta\}}.$$

Consequently, if $F\zeta = F\omega + F\eta$ and $F\varepsilon = F\omega - F\eta$, the relations, furnished by Euler's well-known integration, between $\zeta\varepsilon$ and $\omega\eta$ yield

$$\sqrt{-T}\{\Pi(-\operatorname{cosec}^2\eta) + \Pi(-c^2\sin^2\eta) - F\}(c\omega) = \tfrac{1}{4}\log\frac{\sin^2\zeta}{\sin^2\varepsilon} \quad \ldots(15),$$

in which $\Pi(-\operatorname{cosec}^2\eta)$ and the logarithm both become infinite at the crisis $\omega = \eta$, $\varepsilon = 0$. In future, we set aside the case of parameters negative and greater than unity, as sufficiently disposed of by this equation.

Passing to the circular Π, we may doubly modify the reciprocal equation by supposing p positive or negative. But it will suffice to make p positive, and to treat a negative parameter $(-r)$ as its conjugate. Generally, when T is positive, equation (3) may take the form

$$\sqrt{T}.\{\Pi(p) + \Pi(c^2p^{-1}) - F\} = \tan^{-1}.(\sqrt{T}Q).$$

But when $p = \cot^2\theta$,

$$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\sin\theta_0} \quad \ldots\ldots\ldots\ldots(16),$$

an expression which is very easy to remember: and the corresponding value of $\sqrt{T}(-r)$, the conjugate, is no additional burden to the memory, if we do but remember the relation $\sqrt{T}(p).\sqrt{T}(-r) = b^2$, from equation (7). Hence

$$\sqrt{T}.Q = \frac{\Delta(b\theta)}{\sin\theta\cos\theta}\cdot\frac{\tan\omega}{\Delta(c\omega)} = \frac{1}{\sin\theta\sin\theta_0}\cdot\frac{\cos\omega^\circ}{b\cos\omega} \quad \ldots(17).$$

Again, when we assume $\omega = \tfrac{1}{2}\pi$, $Q = \infty$, $\tan^{-1}(\sqrt{T}Q) = \tfrac{1}{2}\pi$; whence

$$\sqrt{T}\{\Pi_c(p) + \Pi_c(c^2p^{-1}) - F_c\} = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots\ldots(17a).$$

Multiply this by $\frac{F\omega}{F_c}$ and subtract the product from the general integral; therefore

$$\left.\begin{aligned}\sqrt{T}\{P(p)+P(c^2p^{-1})\}\,(c\omega) = \cot^{-1}\left\{\sin\theta\sin\theta_\circ.\frac{b\cos\omega}{\cos\omega^\circ}\right\}\\ -\tfrac{1}{2}\pi.\frac{F(c\omega)}{F_c}\end{aligned}\right\}\ldots(18),$$

when $p=\cot^2\theta$.

VI. From the reciprocal we proceed to the conjugate equation (4).

When Π is logarithmic, p as well as $-r$ will be negative, and we may write $-r'$ for p, so that we get

$$\left.\begin{matrix}R=\sin\omega\sin\omega^\circ\\(1-r)(1-r')=b^2\end{matrix}\right\}\text{ and }\frac{1-r}{r}\Pi(-r)+\frac{1-r'}{r'}\Pi(-r')-\frac{c^2}{rr'}F=\int_0\frac{-dR}{1-rr'.R_2}$$

$$=\frac{-1}{2\sqrt{(rr')}}.\log.\frac{1+R\sqrt{(rr')}}{1-R\sqrt{(rr')}},$$

it being observed that Rrr' are all numerically less than 1.

If $r=c^2\sin^2\eta$, $r'=c^2\sin^2\eta^\circ$; and the logarithmic part is

$$\log\frac{1+c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}{1-c\sin\omega\sin\omega^\circ.\,c\sin\eta\sin\eta^\circ}\text{ or }\log\frac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1},$$

if we form c_1, ω_1, η_1 in Lagrange's scale from c, ω, η.

The same may take another form: viz. if $F\zeta=F\omega+F\eta$ and $F\varepsilon=F\omega-F\eta$, it becomes

$$\log\frac{\Delta\omega\Delta\eta+c^2\sin\omega\cos\omega\sin\eta\cos\eta}{\Delta\omega\Delta\eta-c^2\sin\omega\cos\omega\sin\eta\cos\eta}\text{ or }\log\frac{\Delta\varepsilon}{\Delta\zeta}.$$

Further, observe that

$$\sqrt{(rr')}\,\frac{1-r}{r}=\sqrt{T(-r)},\quad\sqrt{(rr')}\,\frac{1-r'}{r'}=\sqrt{T(-r')},$$

so that the general integral becomes

$$\sqrt{-T}\Pi(-r)+\sqrt{-T}\Pi(-r')=\frac{c^2}{\sqrt{(rr')}}F-\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta}\ldots(19).$$

In this, let $\omega=\frac{1}{2}\pi$, $\omega^\circ=0$, $R=0$, $\varepsilon=-\zeta$, therefore

$$\sqrt{-T}\Pi_c(-r)+\sqrt{-T}\Pi_c(-r')=\frac{c^2}{\sqrt{(rr')}}F_c\ldots\ldots..(19a).$$

Multiply the last by $\dfrac{F\omega}{F_c}$ and subtract the product from (19), therefore

$$\left.\begin{aligned}&\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta)+\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta^\circ)\\&=-\tfrac{1}{2}\log\frac{1+c_1\sin\omega_1\sin\eta_1}{1-c_1\sin\omega_1\sin\eta_1}=-\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta}\end{aligned}\right\}\ldots(20).$$

But the more important case is when Π is circular. Let

$$p=\cot^2\theta,\quad r=-1+b^2\sin^2\theta.$$

Observe that

$$\sqrt{(pr)}=\frac{p}{1+p}\sqrt{T(p)}=\frac{r}{1-r}\sqrt{T(-r)},$$

so that

$$\sqrt{T}.\Pi(p)-\sqrt{T}\Pi(-r)-\frac{c^2}{\sqrt{(pr)}}F=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(21).$$

We now see also that equation (6) admitted of being written

$$\sqrt{T}\{\Pi\omega+\Pi\omega^\circ-\Pi_c\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ)\ldots(22),$$

where p is the common parameter: but the advantage of comparing the two last equations is best seen, when we have separated P out of Π. In (21), make $\omega=\frac{1}{2}\pi$, then

$$\sqrt{T}\Pi_c(p)-\sqrt{T}\Pi_c(-r)-\frac{c^2}{\sqrt{(pr)}}F_c=0.$$

Multiply by $\dfrac{F\omega}{F_c}$ and subtract from (21); then

$$\sqrt{T}\mathrm{P}(p)-\sqrt{T}\mathrm{P}(-r)=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(23).$$

Also subtract from (5) the equation

$$\sqrt{T}\left\{\frac{F\omega}{F_c}+\frac{F\eta}{F_c}-\frac{F\zeta}{F_c}\right\}\quad{}_c=0,$$

and it changes every Π in (5) into P. Observing, then, that $\mathrm{P}_c=0$, we have instead of (22) the simpler result

$$\sqrt{T}\{\mathrm{P}(p\omega)+\mathrm{P}(p\omega^\circ)\}=\tan^{-1}\{\sqrt{(pr)}\sin\omega\sin\omega^\circ\}\ldots(24).$$

Comparing then (23) and (24), we conclude that

$$\sqrt{T}.\mathrm{P}(-r,\omega)=-\sqrt{T}.\mathrm{P}(p,\omega^\circ)\ldots\ldots\ldots\ldots(25).$$

Hence the general enunciation: "In any circular $\sqrt{T}\mathrm{P}$ we may at pleasure change a parameter into its conjugate, provided that we change the amplitude also into the negative of its conjugate."

It is not difficult to verify (25) directly, by mere differentiation. Moreover, if we adapt the process to the case of a logarithmic Π, then, instead of (24), we get

$$\sqrt{-T}\{\mathrm{P}(-r, \omega)+\mathrm{P}(-r, \omega^\circ)\} = -\tfrac{1}{2}\log\frac{\Delta\varepsilon}{\Delta\zeta},$$

which, compared with (20), gives

$$\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta^\circ, \omega) = \sqrt{-T}\mathrm{P}(-c^2\sin^2\eta, \omega^\circ) \ldots (25a).$$

Thus, "In a logarithmic $\sqrt{-T}\mathrm{P}$ we may change the parameter into its conjugate, provided that we simultaneously change the amplitude into its conjugate. Or, "To get the integral conjugate to $\sqrt{-T}\mathrm{P}(-c^2\sin^2\eta, \omega)$, we may at pleasure put η° for η, or ω° for ω."

Generally, even with a circular Π, it suffices to treat of *three* parameters, as p, its reciprocal q, and $-r$ the conjugate to p. But we may reckon *four* in the following method: first, $\mathrm{P}(\cot^2\theta, c, \omega)$ its reciprocal $\mathrm{P}(c^2\tan^2\theta, c, \omega)$; conjugate of the first, $-\mathrm{P}(\cot^2\theta, c, \omega^\circ)$; conjugate of the second, $-\mathrm{P}(c^2\tan^2\theta, c, \omega^\circ)$. And these, though in appearance four, are evidently in form only two. But for the present we shall continue to deal with three.

VII. Let us for conciseness write $\dfrac{F(c\omega)}{F_c} = \dfrac{x}{\frac{1}{2}\pi}$,

$$\Omega = \sqrt{TP(pc\omega)},\quad \overset{2}{\Omega} = \sqrt{TP(qc\omega)},\quad \overset{3}{\Omega} = \sqrt{TP(-rc\omega)},$$

where $p = \cot^2\theta$, $q = c^2\tan^2\theta = \cot^2\theta_\circ$, $r = \Delta^2(b\theta)$;

and there will be no danger of mistaking $\overset{2}{\Omega}$, $\overset{3}{\Omega}$ for *powers* of Ω, since no powers of Ω ever occur in any equation with which we deal.

Equations (18) and (23) now give the two results

$$\cot(x+\Omega+\overset{2}{\Omega}) = \sin\theta\sin\theta_\circ \cdot \frac{b\cos\omega}{\cos\omega^\circ} \ldots\ldots (26),$$

$$\tan(\Omega-\overset{3}{\Omega}) = \sin\omega\sin\omega^\circ \cdot \frac{c\cos\theta}{\cos\theta_\circ} \ldots\ldots\ldots (27);$$

which immediately suggest that if in (27) we commute $c\omega\theta$ into $b\theta\omega$, we make the members of the two equations identical.

Again, we may propose to ourselves to eliminate Ω between (26) and (27), or rather between (18) and (23), so as to get a relation between $\overset{2}{\Omega}$ and $\overset{3}{\Omega}$. This has been done by

Legendre, under a different notation; and the result is most unexpectedly simple.

For a moment, let

$$h^{-1} = \frac{\sin\theta\cos\theta}{\Delta(b\theta)} \cdot \frac{\Delta(c\omega)}{\tan\omega}, \quad k = \frac{\sin\omega\cos\omega}{\Delta(c\omega)} \cdot \frac{\Delta(b\theta)}{\tan\theta};$$

or, if $D = \dfrac{\Delta(b\theta)\sin\omega}{\Delta(c\omega)\sin\theta}$, $h = \dfrac{D}{\cos\omega\cos\theta}$, $k = D\cos\omega\cos\theta$.

Also $\qquad x + \Omega + \overset{2}{\Omega} = \tan^{-1}h : \quad \Omega - \overset{3}{\Omega} = \tan^{-1}k;$

whence $\qquad x + \overset{2}{\Omega} + \overset{3}{\Omega} = \tan^{-1}h - \tan^{-1}k,$

or $\qquad \tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \dfrac{h-k}{1+hk}.$

Now $1 + hk = 1 + D^2 = \dfrac{\Delta^2(c\omega)\sin^2\theta + \Delta^2(b\theta)\sin^2\omega}{\Delta^2(c\omega)\sin^2\theta} = \dfrac{1 - \cos^2\omega\cos^2\theta}{\Delta^2(c\omega)\sin^2\theta}$,

and $\qquad h - k = h(1 - \cos^2\omega\cos^2\theta);$

$\therefore \dfrac{h-k}{1+hk} = h.\Delta^2(c\omega)\sin^2\theta = \dfrac{\Delta(b\theta)}{\cot\theta} \cdot \dfrac{\Delta(c\omega)}{\cot\omega} = \text{also } \dfrac{\sin\theta}{\sin\theta_0} \cdot \dfrac{\sin\omega}{\sin\omega^\circ}.$

Thus we obtain between $\overset{2}{\Omega}$ and $\overset{3}{\Omega}$ the relation

$$\tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \frac{\sin\theta}{\sin\theta_0} \cdot \frac{\sin\omega}{\sin\omega^\circ} \quad \ldots\ldots\ldots (28).$$

The three equations (26), (27), (28) are a mere application of the *Reciprocal* and *Conjugates* of Legendre to the case of a circular $\sqrt{TP}$.

VIII. We now turn to the *Commutative* equations.

Making p to vary in Π, we get $\dfrac{d\Pi}{dp} = \displaystyle\int_0 \frac{-\sin^2\omega . dF}{(1+p\sin^2\omega)^2}$. By the general formula of reduction this receives the shape

$$\frac{d\Pi}{dp} = \alpha . \frac{\sin\omega\cos\omega\Delta\omega}{1+p\sin^2\omega} + \beta . F + \gamma . \int_0 \sin^2\omega\, dF + \delta . \Pi,$$

and it is found that $\alpha, \beta, \gamma, \delta$ have a common denominator, which is none other than $T(p)$. Calling the numerators $\alpha', \beta', \gamma', \delta'$, we find

$$\alpha' = \tfrac{1}{2}, \quad \beta' = -\tfrac{1}{2} . \frac{c^2}{p^2}, \quad \gamma' = -\tfrac{1}{2} . \frac{c^2}{p}, \quad \delta' = -\tfrac{1}{2} . \left(1 - \frac{c^2}{p^2}\right),$$

which farther yield

$$\delta' = -\tfrac{1}{2}\cdot\frac{dT}{dp}, \quad \beta' = \tfrac{1}{2}\cdot\left(\frac{dT}{dp} - 1\right).$$

Observing then that $c^2 \int_0 \sin^2\omega . dF = F - E$, we get

$$T.\frac{d\Pi}{dp} = \tfrac{1}{2}.\frac{\sin\omega\cos\omega\Delta\omega}{1+p\sin^2\omega} + \tfrac{1}{2}.\left(\frac{dT}{dp}-1\right)F - \frac{1}{2p}(F-E) - \tfrac{1}{2}.\frac{dT}{dp}.\Pi.$$

Bringing all the T's and Π's to the left, we divide either by $\sqrt{T}$ or by $-\sqrt{-T}$ to make the left-hand an exact integral, and then integrate nearly as Legendre.

It gives

$$\left.\begin{aligned} &\sqrt{T}(\Pi - F) \\ &= \tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{T^{-1}}dp}{1+p\sin^2\omega} - \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{T}} + \tfrac{1}{2}E\int\frac{dp}{p\sqrt{T}} \\ &\sqrt{-T}(\Pi - F) \\ &= -\tfrac{1}{2}\sin\omega\cos\omega\Delta\omega.\int\frac{\sqrt{-T^{-1}}dp}{1+p\sin^2\omega} + \tfrac{1}{2}F\int\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E\int\frac{dp}{p\sqrt{-T}} \end{aligned}\right\}\text{..(29)},$$

in which the integrals may begin from $p = 0$, since this supposition makes the left-hand member vanish, and also involves no infinite quantities, such as would appear in Legendre's equation.

In the last let $\omega = \tfrac{1}{2}\pi$;

$$\therefore\ \sqrt{-T}(\Pi_c p - F_c) = \tfrac{1}{2}F_c.\int_0\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} - \tfrac{1}{2}E_c\int_0\frac{dp}{p\sqrt{-T}}\text{...(29}a\text{)}.$$

Now, if $\qquad p = -c^2\sin^2\eta, \quad \sqrt{-T} = \cot\eta\Delta(c\eta),$

whence $\tfrac{1}{2}.\left(1+\frac{1}{p}\right)\frac{dp}{\sqrt{-T}} = \Delta(c\eta)\,d\eta, \quad \tfrac{1}{2}.\frac{dp}{p\sqrt{-T}} = \frac{d\eta}{\Delta(c\eta)},$

$$\text{or}\ \left.\begin{aligned}\sqrt{-T}\{\Pi_c(-c^2\sin^2\eta) - F_c\} &= F_cE(c\eta) - E_cF(c\eta) \\ &= F_cG(c\eta)\end{aligned}\right\}\text{...(30)}.$$

We may now apply equation (13) to save the trouble of a second integration for the case of the circular Π. It is only requisite to put $\sin\eta = \sqrt{-1}\tan\theta$, and there follows (multiplying or dividing by $\sqrt{-1}$),

$$\sqrt{T}\{F_c - \Pi_c(c^2\tan^2\theta)\} = -F_c\{H(b\theta) - \Delta(b\theta)\tan\theta\}\text{...(31)}.$$

There is ambiguity as to the sign of $\sqrt{T}$, which is changed according as we *multiply* or *divide* by $\sqrt{-1}$. But we determine the sign by observing that when p is positive, $F > \Pi$; also when θ approaches $\tfrac{1}{2}\pi$, evidently $\Delta(b\theta)\tan\theta > H(b\theta)$, since $F_cH_b = \tfrac{1}{2}\pi$.

We may deduce $\Pi_c(-1+b^2\sin^2\theta)$ by combining the last with (21) and with (17*a*).

From 21,

$$\sqrt{T}\Pi_c(p)-\sqrt{T}\Pi_c(-r)=\frac{c^2}{\sqrt{(pr)}}F_c.$$

Also $\sqrt{T}\{\Pi_c p+\Pi_c q-F_c\}=\frac{1}{2}\pi$ from (17*a*);

therefore $\sqrt{T}\{\Pi_c(q)-F_c\}+\sqrt{T}\Pi_c(-r)=\frac{1}{2}\pi-\frac{c^2}{\sqrt{(pr)}}F_c,$

or $\sqrt{T}\{\Pi_c(-r)-F_c\}-\sqrt{T}\{F_c-\Pi_c(q)\}=\frac{1}{2}\pi-\left\{\sqrt{T}(-r)+\frac{c^2}{\sqrt{(pr)}}\right\}F_c$

$$=\frac{1}{2}\pi-\Delta(b\theta)\tan\theta.F_c,\quad \text{if } p=\cot\theta.$$

Add the last to (31), and it gives

$$\sqrt{T}\{\Pi_c(-\Delta^2.b,\theta)-F_c\}=\frac{1}{2}\pi-F_c.H(b\theta)\ldots(32).$$

This completes the equation needed for the integral Π_c, which is entirely reduced to F and E.

We vary the form only, by adding (17*a*) to (31);

therefore $\sqrt{T}\Pi_c(p)=\frac{1}{2}\pi-F_c\{H(b\theta)-\Delta(b\theta)\tan\theta\}$;

subtract it from $\sqrt{T}.F_c=\frac{\Delta(b\theta)}{\sin\theta\cos\theta}.F_c$;

then $\sqrt{T}\{F_c-\Pi_c(\cot^2\theta)\}=F_c\{H(b\theta)+\Delta(b\theta)\cot\theta\}-\frac{1}{2}\pi\ldots(33).$

We may also develop the right-hand member of equation (30) by Lagrange's scale; then

$\sqrt{-T}.\{\Pi_c(-c^2\sin^2\eta)-C\}=C_1c_1\sin\eta_1+C_2c_2\sin\eta_2+C_3c_3\sin\eta_3+\&c\ldots;$

and if we then assume $-c_n^{\,2}\sin^2\eta_n=p_n$,

$$\left.\begin{aligned}\sqrt{-T}.\{\Pi_c-C\}&=C_1\sqrt{-p_1}+C_2\sqrt{-p_2}+C_3\sqrt{-p_3}+\&c\ldots\\ \therefore\ \sqrt{T}.\{C-\Pi_c\}&=C_1\sqrt{p_1}+C_2\sqrt{p_2}+C_3\sqrt{p_3}+\&c\ldots\end{aligned}\right\}\ldots(30a),$$

if we suppose p positive in the last. But to this subject we shall recur.

Π_c being now fully known, it remains to investigate P only, and Π will be known.

We must return to equation (29). If we multiply (29*a*) by $\frac{F(c\omega)}{F_c}$ and combine it with (29), we get

$$\sqrt{-T}P=-\frac{1}{2}\sin\omega\cos\omega\Delta(c\omega)\int_0\frac{\sqrt{-T^{-1}}dp}{1+p\sin^2\omega}-G(c\omega)\int_0\frac{dp}{2p\sqrt{-T}}.$$

We have already found the last integral $= F(c\eta)$, when $p = -c^2 \sin^2\eta$. Also

$$\frac{\sqrt{-T^{-1}}.\,dp}{1+p\sin^2\omega} = \frac{2c^2 \sin\eta\cos\eta\, d\eta}{(1-c^2\sin^2\omega\sin^2\eta)\cot\eta\,\Delta(c\eta)} = \frac{-2c^2.\sin^2\eta.dF(c\eta)}{1-c^2\sin^2\omega\sin^2\eta}$$

$$= \frac{2}{\sin^2\omega}\left\{1 - \frac{1}{1-c^2\sin^2\omega\sin^2\eta}\right\} dF(c\eta);$$

therefore $\sqrt{-}\,TP(-c^2\sin^2\eta, c, \omega)$

$$= \frac{\Delta(c\omega)}{\tan\omega}\{\Pi(-c^2\sin^2\omega, c, \eta) - F(c\eta)\} - G(c\omega).F(c\eta),$$

where we may write simply $\sqrt{-}\,T$ for the multiplier of the quantity in brackets.

Again, in (30) change η into ω, and multiply by $F(c\eta) \div F_c$; therefore $0 = \sqrt{-}\,T.\{\Pi_c(-c^2\sin^2\omega) - F_c\}\dfrac{F(c\eta)}{F_c} - G(c\omega)F(c\eta)$: subtract this from the preceding; then

$$\sqrt{-}\,TP(-c^2\sin^2\eta, c, \omega) = \sqrt{-}\,TP(-c^2\sin^2\omega, c, \eta)\ldots(34);$$

this is the commutative equation for a logarithmic P.

For a circular P the result is not quite so simple. We have first

$$\sqrt{T}P = \sin\omega\cos\omega\,\Delta(c\omega)\int_0 \frac{\frac{1}{2}\sqrt{T^{-1}}.dp}{1+p\sin^2\omega} + G(c\omega)\int_0 \frac{dp}{2p\sqrt{T}}.$$

Let $p = \cot^2\theta;$

$$\therefore \frac{dp}{2p} = \frac{-d\theta}{\sin\theta\cos\theta}, \quad \sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta}, \quad \frac{dp}{2p\sqrt{T}} = \frac{-d\theta}{\Delta(b\theta)} = -dF(b\theta).$$

Also

$$\frac{1}{2\sqrt{T}}\cdot\frac{dp}{1+p\sin^2\omega} = \frac{-\cot^2\theta\, d\theta}{\Delta(b\theta)(1+\sin^2\omega\cot^2\theta)} = \frac{-\cos^2\theta.dF(b\theta)}{\sin^2\theta+\sin^2\omega.\cos^2\theta}.$$

The denominator $= \sin^2\omega + \cos^2\omega\sin^2\theta = \sin^2\omega(1+\cot^2\omega\sin^2\theta)$;

$$\therefore \int\frac{1}{2\sqrt{T}}\cdot\frac{dp}{1+p\sin^2\omega} = \frac{F(b\theta)}{\cos^2\omega} - \frac{\Pi(\cot^2\omega, b, \theta)}{\sin^2\omega\cos^2\omega} + \text{const.}$$

Instead of correcting by making $\theta = \frac{1}{2}\pi$, $p = 0$, we may make $p = \infty$, $\theta = 0$; in which case we know that $\sqrt{T}\Pi(p) = \frac{1}{2}\pi$ for all values of ω, and consequently

$$\sqrt{T}P(p, c, \omega) = \tfrac{1}{2}\pi\left\{1 - \frac{F(c\omega)}{F_c}\right\} = \tfrac{1}{2}\pi - x;$$

$$\therefore \sqrt{T}P(\cot^2\theta, c, \omega) = \sin\omega\cos\omega\,\Delta(c\omega)\left\{\frac{F(b\theta)}{\cos^2\omega} - \frac{\Pi(\cot^2\omega, b, \theta)}{\sin^2\omega\cos^2\omega}\right\}$$
$$- G(c\omega)F(b\theta) + (\tfrac{1}{2}\pi - x).$$

Now put $\theta = \frac{1}{2}\pi$, and the left-hand member vanishes;

$$\text{or } 0 = \sin\omega\cos\omega\Delta(c\omega)\left\{\frac{F_b}{\cos^2\omega} - \frac{\Pi_b(\cot^2\omega)}{\sin^2\omega\cos^2\omega}\right\} - G(c\omega)F_b + (\tfrac{1}{2}\pi - x).$$

Multiply the last by $\dfrac{F(b\theta)}{F_b} = \dfrac{t}{\frac{1}{2}\pi}$, and subtract from the penultimate; observing that $\dfrac{\Delta(c\omega)}{\sin\omega\cos\omega} = \sqrt{T}(\cot^2\omega)$; hence

$$\sqrt{T}P(\cot^2\theta, c, \omega) + \sqrt{T}P(\cot^2\omega, b, \theta) = (\tfrac{1}{2}\pi - x)\left(1 - \frac{t}{\frac{1}{2}\pi}\right) \ldots (35),$$

in which the right-hand member could not have been conjectured from the analogy of (34). It has risen from the peculiarity that $\sqrt{T}\Pi(p) = \frac{1}{2}\pi$ when $p = \infty$.

IX. We may now revert to the notation of (26), (27), (28), by writing Θ for that which Ω becomes when $\theta c\omega$ are changed to $\omega b\theta$. Then, in place of (35), we get

$$\tfrac{1}{2}\pi\left\{\Omega + \Theta\right\} = (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - t) \ldots\ldots\ldots (36).$$

Between the quantities $\Omega, \overset{2}{\Omega}, \overset{3}{\Omega}, \Theta, \overset{2}{\Theta}, \overset{3}{\Theta}$ we have obtained three independent equations (26), (27), (36). If we commute θ, c, ω with ω, b, θ, (36) is not changed, but (26) and (27) yield two new equations, so that we have, in all, *five* equations connecting the *six* quantities in pairs. We have, therefore, exhausted all the relations between them; and by mere elimination we can obtain the equation between any pair of them at pleasure.

In a single view the equations are as follows:

$$\left.\begin{aligned} &\cot(x + \Omega + \overset{2}{\Omega}) = \tan(\Theta - \overset{3}{\Theta}) \\ &= \tan\left\{\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\Omega}\right\} = \tan\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \Omega - \overset{3}{\Theta}\right\} \\ &= \sin\theta\sin\theta_0 \cdot \frac{b\cos\omega}{\cos\omega^\circ} \end{aligned}\right\} \ldots (37),$$

in which we may exchange $\theta, c, \omega, \Omega, x$ with $\omega, b, \theta, \Theta, t$.

$$\left.\begin{aligned} &\text{Again, } \tan(x + \overset{2}{\Omega} + \overset{3}{\Omega}) = \tan(t + \overset{2}{\Theta} + \overset{3}{\Theta}) \\ &= \tan\left\{\frac{xt}{\frac{1}{2}\pi} + \overset{2}{\Omega} + \overset{2}{\Theta}\right\} = \cot\left\{\frac{(\frac{1}{2}\pi - x)(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} - \overset{3}{\Omega} - \overset{3}{\Theta}\right\} \\ &= \frac{\sin\theta}{\sin\theta_0} \cdot \frac{\sin\omega}{\sin\omega^\circ} \end{aligned}\right\} \ldots (38).$$

Lastly,

$$\left.\begin{aligned} \tfrac{1}{2}\pi(\Omega+\Theta) &= (\tfrac{1}{2}\pi - x)(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi(\overset{2}{\Theta} - \overset{3}{\Omega}) &= x(\tfrac{1}{2}\pi - t) \\ \tfrac{1}{2}\pi(\overset{2}{\Omega} - \overset{3}{\Theta}) &= t(\tfrac{1}{2}\pi - x) \end{aligned}\right\} \text{.........(39).}$$

In (37) there are, with the commutations, 8 equations; in (38) there are 4; so that we have here 15 relations of pairs of the six integrals. Also $15 = \frac{5.6}{2}$; so that all the possible pairs are here exhibited.

Thus by a single transformation we can pass from one circular $\sqrt{TP}$ to another, so as to secure at once that p shall be less than q, (and therefore, p less than c), and that the modulus shall be less than its complement. We may also, if we wish it, secure a positive parameter.

X. Now rises the question, of approximating to $\sqrt{TP}$, when we have so transformed it as to obtain a favourable case. We proceed by Lagrange's scale, supposing $c_1\omega_1$ to be deduced as usual from $c\omega$. Let p also be given, but p_1r be disposable constants. Then if Π_1 stand for $\Pi(p_1c_1\omega_1)$, and we make the usual substitutions $F(c_1\omega_1) = (1+b)F(c\omega)$, $\sin\omega_1 = (1+b)\frac{\sin\omega\cos\omega}{\Delta(c\omega)}$, we get

$$\Pi_1 = \int_0 \frac{dF(c_1\omega_1)}{1+p_1\sin^2\omega_1} = (1+b)\int_0 \frac{\Delta^2(c\omega)\,dF(c\omega)}{\Delta^2(c\omega)+p_1(1+b)^2\sin^2\omega\cos^2\omega}.$$

We determine the relations between p_1, p, and r, if we assume the denominator to be $=(1+p\sin^2\omega)(1-r\sin^2\omega)$; which gives $pr = p_1(1+b)^2$, and making $\omega = \frac{1}{2}\pi$, $(1+p)(1-r) = b^2$; so that $-r$ is none other than the parameter conjugate to p.

Also,
$$p_1 = \frac{pr}{(1+b)^2}, \quad q_1 = \frac{(1-b)^2}{pr}.$$

Hence
$$T_1 = (1+p_1)(1+q_1) = \frac{(\overline{1+b}|^2+pr)(\overline{1-b}|^2+pr)}{(1+b)^2.pr},$$

of which the numerator

$$= c^4 + 2(1+b^2)pr + p^2r^2 = (pr-c^2)^2 + 4pr = (p-r)^2 + 4pr = (p+r)^2;$$

therefore
$$\sqrt{\frac{T_1}{pr}} = \frac{p+r}{(1+b)pr}.$$

Thus T_1 is positive, when pr is positive, or T_1 T are both circular or both logarithmic.

Farther, if we assume

$$\frac{1-c^2v}{(1+pv)(1-rv)}=\frac{M}{1+pv}+\frac{N}{1-rv},$$

which yields

$$\Pi_1=(1+b)\{M\Pi(p)+N\Pi(-r)\},$$

we get $M+N=1$, $Mr-Np=c^2$; whence

$$M=\frac{p+c^2}{p+r}=\frac{1+p}{p}\cdot\left(\frac{pr}{p+r}\right),\ \text{and}\ N=\frac{r-c^2}{p+r}=\frac{1-r}{r}\cdot\left(\frac{pr}{p+r}\right),$$

whence $\Pi_1=(1+b)\cdot\dfrac{pr}{p+r}\left\{\dfrac{1+p}{p}\Pi(p)+\dfrac{1-r}{r}\Pi(-r)\right\}$.

Multiply by $\sqrt{T_1}=\dfrac{p+r}{(1+b)\sqrt{(pr)}}$;

therefore $\sqrt{T_1}\,\Pi_1=\sqrt{T}\,\Pi(p)+\sqrt{T}\,\Pi(-r)\ldots\ldots\ldots(40).$

Make $\omega=\frac{1}{2}\pi$, $\omega_1=\pi$; $\therefore\ 2\sqrt{T_1}\Pi_{c1}=\sqrt{T}\Pi_c(p)+\sqrt{T}\Pi_c(-r)$.

Multiply this by $\frac{1}{2}\cdot\dfrac{F_1}{C_1}=\dfrac{F}{C}$, and subtract from (40),

then $\sqrt{\pm\,T_1}\mathrm{P}_1=\sqrt{\pm\,T}\mathrm{P}(p)+\sqrt{\pm\,T}\mathrm{P}(-r)\ldots\ldots(40a)$,

which for the circular integral is

$$\Omega_1=\Omega+\overset{s}{\Omega}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41).$$

But from (27), observing that

$$\sin\omega\,\sin\omega^\circ\cdot\frac{c\cos\theta}{\cos\theta_0}=\frac{\sin\omega_1}{1+b}\cdot\sqrt{(pr)}=\sqrt{p_1}\,\sin\omega_1,$$

we have $\Omega-\overset{s}{\Omega}=\tan^{-1}(\sqrt{p_1}\sin\omega_1)$.

Add this to (41), and you get

$$\Omega=\tfrac{1}{2}\Omega_1+\tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1)\ldots\ldots\ldots\ldots(42):$$

but in the process there has been nothing up to (40*a*) to limit p to be circular. When otherwise, it is very obvious how to change $\tan^{-1}$ into $\sqrt{-1}.\log$ in the equation (42) so as to adapt it as in equation (20).

XI. Legendre has investigated the relations of p_1 and p; but we need to add many reflections.

When $p=-c^2\sin^2\eta$, $r=c^2\sin^2\eta^\circ$,

$$\sqrt{(-pr)}=c^2\sin\eta\,\sin\eta^\circ=\frac{c^2}{1+b}\sin\eta_1;$$

therefore $$\sqrt{-p_1} = \frac{c^2}{(1+b)^2} \sin\eta_1 = c_1 \sin\eta_1.$$

Consequently the series p, p_1, p_2...answers the conditions supposed in equation (30*a*), and those developments apply to our newly-derived parameters.

When we assume $\sin\eta = \sqrt{-1} \tan\theta$, or $p = c^2 \tan^2\theta$, the general equation of Lagrange's scale, viz. $\tan\eta_1 = \frac{(1+b)\tan\eta}{1 - b\tan^2\eta}$, changes into the scale of Gauss, $\sin\theta_1 = \frac{(1+b)\sin\theta}{1+b\sin^2\theta}$, which is that followed by $\theta, \theta_1, \theta_2, \theta_3$... Consequently, when $\theta = \frac{1}{2}\pi$, every other θ in the scale also $= \frac{1}{2}\pi$. Also $\theta, \theta_1, \theta_2, \theta_3$... tend to $\frac{1}{2}\pi$ as their limit, since they give

$$\frac{F(b\theta)}{B} = \frac{F(b_1\theta_1)}{B_1} = \frac{F(b_2\theta_2)}{B_2};$$

and since $B_n = \infty$ when $n = \infty$, and $b_n = 1$, therefore

$$F(1\theta_n) = \infty, \text{ or } \theta_n = \tfrac{1}{2}\pi.$$

If we form $\theta, \theta', \theta''$... in the opposite direction, then, since

$$\frac{t}{\frac{1}{2}\pi} = \frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'} = \frac{F(b''\theta'')}{B''} = \&c...$$

and $B^{(n)}$ has $\frac{1}{2}\pi$ for limit, and $b^{(n)}$ is evanescent, therefore $\theta^{(n)} = t$, when $n = \infty$; or $\theta, \theta', \theta'', \theta'''$... tend to t as their limit.

It is farther important to observe, that the change of $b\theta$ to $b_n\theta_n$ or to $b^{(n)}\theta^{(n)}$ leaves t wholly unchanged; inasmuch as

$$\frac{t}{\frac{1}{2}\pi} = \frac{F(b\theta)}{B} = \frac{F(b_n\theta_n)}{B_n} = \frac{F(b^{(n)}\theta^{(n)})}{B^{(n)}}.$$

On the other hand, it is familiar, as a result of the equation in Lagrange's scale,

$$\frac{x}{\frac{1}{2}\pi} = \frac{F(c\omega)}{C} = \tfrac{1}{2}\cdot\frac{F(c_1\omega_1)}{C_1} = \tfrac{1}{4}\cdot\frac{F(c_2\omega_2)}{C_2} = \&c...,$$

that to change $c\omega$ into $c_1\omega_1$ changes x into $2x$, &c.

The equation $\sin\theta_1 = \frac{(1+b)\sin\theta}{1+b\sin^2\theta}$, which gives $p_1 = c_1^2 \tan^2\theta_1$, admits of other forms; especially

$$\cos\theta_1 = \frac{\cos\theta.\Delta(b\theta)}{1+b\sin^2\theta}, \quad \Delta(b_1\theta_1) = \frac{1-b\sin^2\theta}{1+b\sin^2\theta};$$

which last gives inversely

$$b \sin^2\theta = \frac{1 - \Delta(b_1\theta_1)}{1 + \Delta(b_1\theta_1)}.$$

Legendre calculates $p, p_1, p_2 \ldots$ by auxiliary arcs $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$ Let $\cos\lambda = \Delta(b\theta)$, $\cos\lambda_1 = \Delta(b_1\theta_1)$, &c.; therefore $\sin\lambda = b\sin\theta$. Consequently $\frac{\sin^2\lambda}{b} = \frac{1 - \cos\lambda_1}{1 + \cos\lambda_1}$, or $\sin\lambda = \sqrt{b} \,.\, \tan\frac{1}{2}\lambda_1$; which is a general relation for $\lambda, \lambda_1, \lambda_2, \lambda_3 \ldots$

The arcs $\lambda, \lambda_1, \lambda_2 \ldots$ are thus suggested by the parameter $\cot^2\theta$, but they equally apply to logarithmic parameters. In fact, if we assume

$$\sin\lambda = \frac{b}{\Delta(c\eta)}, \quad \sin\lambda_1 = \frac{b_1}{\Delta(c_1\eta_1)},$$

and observe that $\Delta^2(c\eta) = b^2 + c^2\cos^2\eta$, we get

$$\cos\lambda = \frac{c\cos\eta}{\Delta(c\eta)}, \quad \cos\lambda_1 = \frac{c_1\cos\eta_1}{\Delta(c_1\eta_1)}.$$

Now, by the known relations in Lagrange's scale,

$$\cos\eta_1 = \frac{1 - (1 + b)\sin^2\eta}{\Delta(c\eta)}, \quad \text{and} \quad \Delta(c_1\eta_1) = \frac{1 - (1 - b)\sin^2\eta}{\Delta(c\eta)},$$

and

$$c_1 = \frac{1 - b}{1 + b};$$

whence

$$\cos\lambda_1 = \frac{1 - b}{1 + b} \cdot \frac{1 - (1 + b)\sin^2\eta}{1 - (1 - b)\sin^2\eta} = \frac{1 - b - c^2\sin^2\eta}{1 + b - c^2\sin^2\eta},$$

and

$$\frac{1 - \cos\lambda_1}{1 + \cos\lambda_1} = \frac{b}{1 - c^2\sin^2\eta}, \quad \text{or} \quad \tan\tfrac{1}{2}\lambda_1 = \frac{\sin\lambda}{\sqrt{b}},$$

as before. Nevertheless, if λ has the same value in both cases, the relation of θ to η is no longer $\sin^2\eta = -\tan^2\theta$, but is $c^2\sin^2\eta = -\cot^2\theta$.

It thus appears that the series $p_1, p_2, p_3 \ldots$ which are derived from p by the law $p_1 = \frac{pr}{(1 + b)^2}$, are calculable for both kinds of integral by the auxiliaries $p = \cot^2\theta = -c^2\sin^2\eta$,

$$\tan\tfrac{1}{2}\lambda_1 = \frac{\sqrt{b}}{\sqrt{(1 + p)}}, \quad \tan\tfrac{1}{2}\lambda_2 = \frac{\sin\lambda_1}{b_1}, \quad \tan\tfrac{1}{2}\lambda_3 = \frac{\sin\lambda_2}{b_2}, \text{ \&c.}\ldots$$

provided only that $1 + p$ be positive.

Put for a moment $\cos\lambda_r = x$, $\cos\lambda_{r+1} = y$; $\therefore \frac{1 - y}{1 + y} = \frac{1 - x^2}{b_r}$; and since $b_1, b_2, b_3 \ldots$ rapidly tend to 1, the last equation tends to give $y = \dfrac{x^2}{2 - x^2}$, or scarcely more than $y = \frac{1}{2}x^2$. Thus $\cos\lambda_1$, $\cos\lambda_2$, $\cos\lambda_3$... soon tend rapidly to zero.

These arcs enable us to embrace in one expression the two series (30*a*).

Let N for a moment stand for the complete integral $\int_0^{\frac{1}{2}\pi} \dfrac{(1+p)\sin^2\omega}{1+p\sin^2\omega} . dF(c\omega)$, so that $\dfrac{pN}{1+p} = F_c - \Pi_c$; and when $p = -c^2\sin^2\eta$, $\sqrt{-T} = \cot\eta\, \Delta(c\eta)$, and

$$\sqrt{-T}\{\Pi_c - F_c\} = \frac{-p}{1+p}.\cot\eta\,\Delta(c\eta).N = \frac{c^2\sin\eta\cos\eta}{\Delta(c\eta)}.N = (1-b)\sin\eta_1.N.$$

Consequently, from equation (30) we deduce two forms,

$$(1-b)\sin\eta_1.N = CG(c\eta); \quad \text{and} \quad \frac{c_1^2\sin\eta_1\cos\eta_1}{\Delta(c_1\eta_1)}.N_1 = C_1G(c_1\eta_1):$$

but $CG(c\eta) - C_1G(c_1\eta_1) = C_1c_1\sin\eta_1$. Combine these, and observe that $(1-b)\,C = 2C_1c_1$; and you get

$$\frac{N}{C} - \tfrac{1}{2}\cos\lambda_1 . \frac{N_1}{C_1} = \tfrac{1}{2} \ldots\ldots\ldots\ldots\ldots\ldots (43),$$

which now applies alike to both sorts of integrals, and easily gives (when $1+p$ is positive)

$$N = \int_0^{\frac{1}{2}\pi} \frac{(1+p)\sin^2\omega}{1+p\sin^2\omega}\, dF$$

$$= C\{\tfrac{1}{2} + \tfrac{1}{4}\cos\lambda_1 + \tfrac{1}{8}\cos\lambda_1\cos\lambda_2 + \tfrac{1}{16}\cos\lambda_1\cos\lambda_2\cos\lambda_3 + \&c\ldots\}\ldots(43a),$$

which is a series of Legendre's slightly simplified.

XII. Write ψ instead of θ_0, so that $\psi, \psi_1, \psi_2, \psi_3\ldots$ may be (relatively to $c, c_1, c_2, c_3\ldots$) the lower conjugates to $\theta, \theta_1, \theta_2, \theta_3\ldots$ Then if $p_n q_n = c_n^2$, we have simultaneously $p_n = c_n^2\tan^2\theta_n = \cot^2\psi_n$ and $q_n = c_n^2\tan^2\psi_n = \cot^2\theta_n$. Now, *I say*, $\psi, \psi_1, \psi_2, \psi_3\ldots$ are related to one another by the scale of Gauss, exactly as are $\theta, \theta_1, \theta_2, \theta_3\ldots$

For we have, by definition,

$$F(b\theta) + F(b\psi) = B \text{ and } F(b_1\theta_1) + F(b_1\psi_1) = B_1:$$

also, by the property of $\theta\theta_1$, we have

$$\frac{F(b\theta)}{B} = \frac{F(b_1\theta_1)}{B_1}; \text{ that is } \left\{1 - \frac{F(b\psi)}{B}\right\} = \left\{1 - \frac{F(b_1\psi_1)}{B_1}\right\},$$

or $\dfrac{F(b\psi)}{B} = \dfrac{F(b_1\psi_1)}{B_1}$:

which proves $\psi\psi_1$ to be related by the scale of Gauss.

Consequently, whether we assume $p = c^2 \tan^2\theta$, $p_1 = c_1^2 \tan^2\theta_1$, &c....; or, on the other hand, assume $p = \cot^2\theta$, $p_1 = \cot^2\theta_1$, &c.... the series $\theta, \theta_1, \theta_2 ...$ which determine $p, p_1, p_2 ...$ are deduced by the very same law. It amounts to the same thing to remark, that we have

$$p_1 = \frac{p^2}{(1+b)^2} \cdot \frac{1+q}{1+p}; \quad q_1 = \frac{q^2}{(1+b)^2} \cdot \frac{1+p}{1+q};$$

so that q_1 is formed from q, by the same law as p_1 from p.

Thus, "When two original parameters are reciprocals, in reference to the original modulus, so are every derived pair, in reference to the new modulus."

It also follows, that the more rapid the convergence of $p, p_1, p_2, p_3, p_4 ...$, the slower is the convergence of $q, q_1, q_2, q_3 ...$ if indeed they converge at all. This makes it important whether p or q be selected to approximate from. Legendre's equation is

$$p_1 = \frac{p}{1+p} \cdot \frac{p+c^2}{(1+b)^2};$$

whence, if $p = ec$, and $p_1 = e_1 c_1$, there follows

$$\frac{e_1}{e} = \frac{e+c}{1+ec}, \text{ or } 1 \pm \frac{e_1}{e} = \frac{(1 \pm c)(1 \pm e)}{1+ec}.$$

If then $p^2 < c^2$, $e^2 < 1$, and we deduce that $\left(1 \pm \frac{e_1}{e}\right)$ is positive, or e_1 is numerically less than e. Hence if $p^2 < c^2$, $p, p_1, p_2, p_3 ...$ decrease more rapidly than $c, c_1, c_2, c_3 ...$

The relation of p_1 to p may also be written

$$\sqrt{p_1} = \frac{p}{1+p} \cdot \frac{\sqrt{T}}{1+b};$$

but our original equation, $\sqrt{p_1} = \frac{\sqrt{(pr)}}{1+b}$, with equal clearness shews that p_1 is positive whenever p is circular. Nor only so. It also denotes that if the original integral be $\Pi(-r, c, \omega)$, the value of p_1 is not altered; for to change p to $-r$, does but change r to $-p$, and leaves (pr) unchanged. Nevertheless, a consideration of the process which elicited equation (42), shews that the conjugates $p, -r$ in the first step downwards generate $+\sqrt{p_1}$ and $-\sqrt{p_1}$ differing in sign: but in the second step they coincide, and both produce the same $\sqrt{p_2}$.

When the problem is reversed, and we desire from p_1 to determine p, it is evident that there are two roots, p and $-r$, *positive* and *negative*, when Π is circular. But we cannot carry the series backward from $-r$, without falling on imaginary parameters. In fact, as p_1 is positive, if it proceed from a real p and r, so $-r$, being negative, cannot proceed from a real p' and r'. It may here deserve remark, that we thus learn of *certain* imaginary parameters, whose integral can be reduced to *one* real Π.

XIII. It remains to develop the actual series.

Let $\Psi = \tan^{-1}(p \sin\omega)$, so that

$$\Omega = \tfrac{1}{2}\Omega_1 + \tfrac{1}{2}\Psi_1.$$

When p is evanescent, $\Pi_e = F_e$, and P becomes identical with $\Pi - F$. We have seen that $\sqrt{T}(\Pi - F)$ vanishes with p; so therefore does $\sqrt{T}\text{P}$. If then $p, p_1, p_2, p_3 \ldots$ decrease beyond all limit, so do $\Omega, \Omega_1, \Omega_2, \Omega_3 \ldots$; and much more does $2^{-n}\Omega_n$ vanish when $n = \infty$. Now, by repetition of the formula,

$$\Omega - 2^{-n}\Omega_n = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \ldots + 2^{-n}\Psi_n;$$

whence $$\Omega = \tfrac{1}{2}\Psi_1 + \tfrac{1}{4}\Psi_2 + \tfrac{1}{8}\Psi_3 + \text{\&c. \&c.}\ldots,$$

which is the final development by *descending* moduli.

In the other notation we have

$$\left.\begin{aligned} \sqrt{T}\text{P}(p, c, \omega) &= \tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) \\ &\quad + \tfrac{1}{8}\tan^{-1}(\sqrt{p_3}\sin\omega_3) + \text{\&c.} \\ \sqrt{T}\text{P}(-r, c, \omega) &= -\tfrac{1}{2}\tan^{-1}(\sqrt{p_1}\sin\omega_1) + \tfrac{1}{4}\tan^{-1}(\sqrt{p_2}\sin\omega_2) \\ &\quad + \tfrac{1}{8}\tan^{-1}.\ \text{\&c.}\ldots \end{aligned}\right\}\ldots(44).$$

Taking the difference of these,

$$\Omega - \overset{3}{\Omega} = \tan^{-1}(\sqrt{p_1}\sin\omega_1);$$

which agrees with (27).

The worst convergence is when $p = c$: and since we may select of c and b the smaller, by means of the commutative equation, the most unfavourable case which needs to be encountered, is, when $p = c = b$. Even then, the series (44) is computable with very moderate trouble. I conclude therefore that it is really sufficient for practical purposes.

Nevertheless, when c is near to 1, we may seek for a development by means of *ascending* moduli. We shall select $\overset{3}{\Theta} = \sqrt{T}\text{P}(b^2\tan^2\omega, b, \theta)$ to calculate, because, when b, ω, θ change to $b^{(n)}, \omega^{(n)}, \theta^{(n)}$, $\overset{3}{\Theta}{}^{(n)}$ vanishes with $b^{(n)}$. In fact it will presently appear that $2^n.\overset{3}{\Theta}{}^{(n)}$ vanishes, when $n = \infty$.

One of the equations marked (37) is

$$\frac{t(\frac{1}{2}\pi - x)}{\frac{1}{2}\pi} + \Theta - \overset{2}{\Omega} = \tan^{-1}\left\{\sin\theta\sin\theta_{\circ}.\frac{b\cos\omega}{\cos\omega^{\circ}}\right\}.$$

Change ω, c, θ, x, t to θ, b, ω, t, x; therefore

$$\frac{x(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega - \overset{2}{\Theta} = \tan^{-1}\{\sqrt{p_1}\sin\omega_1\} = \Psi_1.$$

In the last, change c, ω, θ to c', ω', θ', which changes x to $\frac{1}{2}x$, but leaves t unchanged; therefore

$$\frac{\frac{1}{2}x.(\frac{1}{2}\pi - t)}{\frac{1}{2}\pi} + \Omega' - \overset{2}{\Theta}' = \Psi.$$

Eliminate x and t; then

$$(2\Omega' - \Omega) - (2\overset{2}{\Theta}' - \overset{2}{\Theta}) = 2\Psi - \Psi_1.$$

But from (42) we have $2\Omega' - \Omega = \Psi$.

Hence $$\overset{2}{\Theta} - 2\overset{2}{\Theta}' = (\Psi - \Psi_1)\dots\dots\dots\dots\dots\dots(45),$$

which is our new equation of reduction.

Repeating it n times, we obtain

$$\overset{2}{\Theta} - 2^n.\overset{2}{\Theta}^{(n)} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \dots \text{ to } n \text{ terms.}$$

We now need a Lemma, to prove that $2^n.\overset{2}{\Theta}^{(n)}$ vanishes when $n = \infty$.

First, observe that when c is so small that c^4 and $c^4\tan^4\theta$ are omissible, and $c^2\tan^2\theta$ is the parameter, we have the following values:

$$\Pi(c^2\tan^2\theta, c, \omega) = \int_0(1 - c^2\tan^2\theta\sin^2\omega)(1 + \tfrac{1}{2}c^2\sin^2\omega)\,d\omega,$$
$$= \omega - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})(\omega - \tfrac{1}{2}\sin 2\omega),$$
$$\Pi_c(c^2\tan^2\theta) = \tfrac{1}{2}\pi\{1 - \tfrac{1}{2}c^2(\tan^2\theta - \tfrac{1}{2})\},$$
$$F(c\omega) = \omega + \tfrac{1}{4}c^2(\omega - \tfrac{1}{4}\sin 2\omega),$$
$$F_c = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}c^2):$$

and combining these, we get

$$\text{P} = \tfrac{1}{4}c^2\tan^2\theta\sin 2\omega = \tfrac{1}{4}p\sin 2\omega.$$

Also $$\sqrt{T} = \frac{\Delta(b\theta)}{\sin\theta\cos\theta},$$

which converges to $\frac{1}{\sin\theta}$, therefore

$$\sqrt{T}\text{P} = \tfrac{1}{4}\frac{c^2\tan^2\theta}{\sin\theta}.\sin 2\omega.$$

Similarly then $\overset{2}{\Theta}$, when b is very small, converges to

$$\tfrac{1}{4} \cdot \frac{b^2 \tan^2\omega}{\sin\omega} \cdot \sin 2\theta:$$

and when we change b, θ, ω into $b^{(n)}, \theta^{(n)}, \omega^{(n)}$, we know that $\theta^{(n)}, \omega^{(n)}$ approach to fixed limits t and $\grave{\omega}$ (which are less than $\frac{1}{2}\pi$, if θ, ω are less), so that we have nearly

$$\overset{2}{\Theta}{}^{(n)} = \tfrac{1}{4} b^{(n)2} \cdot \frac{\tan^2\grave{\omega}}{\sin\grave{\omega}} \cdot \sin 2t = \text{k}.b^{(n)2};$$

the quantity k being finite and independent of n. Hence

$$2^n.\overset{2}{\Theta}{}^{(n)} = \text{k}.\{2^n.b^{(n)2}\}.$$

But when $n = \infty$, $2^n.b^{(n)2}$ is evanescent; and indeed the quantity is extremely small when $n = 3$ or even $n = 2$, if c is very near to 1. Hence we get

$$\overset{2}{\Theta} = (\Psi - \Psi_1) + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \&\text{c.}\ldots (46),$$

a series rapidly converging.

When $\overset{2}{\Theta}$ is known, we find Ω or $\overset{2}{\Omega}$ by one of the commutative equations. Thus

$$\Omega = \overset{2}{\Theta} + \Psi_1 - \frac{x}{\frac{1}{2}\pi}\left(\tfrac{1}{2}\pi - t\right)$$

$$= \left\{\Psi - x\left(1 - \frac{t}{\frac{1}{2}\pi}\right)\right\} + 2(\Psi' - \Psi) + 2^2(\Psi'' - \Psi') + \&\text{c.}\ldots$$

Θ has necessarily a form of development similar to (46); for as the parameter of Θ is $\cot^2\omega$, which $= b^2 \tan^2\omega^\circ$, we need only proceed as if ω°, not ω, had been the original amplitude in Ω, and the result of (46) will be Θ instead of $\overset{2}{\Theta}$. To change ω into ω° does not alter $\sin\omega_1$, nor therefore Ψ_1; but it alters $\Psi, \Psi', \Psi''\ldots$, say into $\Psi^\circ, \Psi'^\circ, \Psi''^\circ\ldots$, therefore

$$\Theta = (\Psi^\circ - \Psi_1) + 2(\Psi'^\circ - \Psi^\circ) + 2^2(\Psi''^\circ - \Psi'^\circ) + \&\text{c.}\ldots (46a).$$

Whether this series or the preceding converges better, seems to depend on the evanescence of $2^n\overset{2}{\Theta}{}^{(n)}$ and $2^n\Theta^{(n)}$; *i.e.* on the magnitude of $\frac{\tan^2\omega}{\sin\omega}$, which evidently increases with ω. Hence, if $\omega > \omega^\circ$, (46) seems not to converge so well as (46a); but it converges better, if $\omega < \omega^\circ$. Nevertheless, both converge well, when c is near to 1.

It may farther be observed, that since $\Omega - \Omega' = \Omega' - \Psi$, we have also $\Omega^{(n)} - \Omega^{(n+1)} = \Omega^{(n+1)} - \Psi^{(n)}$. Also since $c, c', c'', c'''\ldots$ $\omega, \omega', \omega'', \omega'''\ldots\theta, \theta', \theta''\ldots$ all tend to fixed limits $1, \grave{\omega}, t$, so do $\Omega', \Omega'', \Omega'''\ldots.\ \Psi', \Psi'', \Psi'''\ldots$ tend to fixed limits $`\Omega`\Psi$; and since $\Omega^{(n)} - \Omega^{(n+1)}$ has limit zero, so has $\Omega^{(n+1)} - \Psi^{(n)}$; *i.e.* $`\Omega = `\Psi$;

or $$\sqrt{T}P(p`, 1, \grave{\omega}) = \tan^{-1}(\sqrt{p}\ \sin\grave{\omega});$$

or, since $p`\grave{\omega}$ are mutually independent, we have generally

$$\sqrt{T}P(p, 1, \omega) = \tan^{-1}(\sqrt{p}\sin\omega)\ldots\ldots\ldots\ldots(47),$$

which may be easily confirmed by direct integration.

Thus in equation (42), as also in the reciprocal, and in the conjugate equation, the function *tan*$^{-1}$ may be replaced by an integral of the form $\sqrt{T}P(\cot^2\mu, 1, \psi)$. In equation (8) we might similarly substitute

$$CG(c\omega) - C_1G(c_1\omega_1) = C_1c_1G(1, \omega_1).$$

It is remarkable how the function $\tan^{-1}(\sqrt{p_1}\sin\omega_1)$ derived according to Lagrange's and Gauss's scale seems to intrude itself into more elementary equations, as (21), (24).

Finally, it will here be remarked that equation (46) is only in appearance an equation of *ascending* moduli; for though $c, c', c''\ldots$ascend, yet b is the modulus of $\overset{2}{\Theta}$; and $b, b', b'', b'''\ldots$ *decrease* by the same law as $c, c_1, c_2\ldots$ Nevertheless, the mode in which $\theta, \theta', \theta''\ldots\omega, \omega', \omega''\ldots$ are derived, is that which we understand to belong to ascending moduli.

XIV. A similar treatment would manifestly apply to the logarithmic P; but Legendre's adaptation of Jacobi's great discovery here supersedes equation (42), by resolving P into a simpler integral. Indeed, Legendre's reduction of Π, or rather of $\sqrt{T}(\Pi - F)$, to the integral $\Upsilon = \int_0 EdF$, will be well exchanged into a reduction of $\sqrt{T}P$ to the integral $V = \int_0 GdF$. Of course, as $G = E - \dfrac{E_c}{F_c}F$, so $V = \Upsilon - \frac{1}{2}\dfrac{E_c}{F_c}F^2$: and as G is the *fluctuant* to E, and P to Π, so is V to E. The process will then be as follows:

By Euler's integration, if $F\omega + F\eta = F\zeta$,

$$E\omega + E\eta - E\zeta = c^2\sin\omega\sin\eta\sin\zeta,$$

and $$\sin\zeta = \frac{\sin\omega\cos\eta\Delta\eta + \sin\eta\cos\omega\Delta\omega}{1 - c^2\sin^2\omega\sin^2\eta}.$$

When η becomes $-\eta$, let ζ become ϵ. Observe that we equally have

$$G\omega + G\eta - G\zeta = c^2\sin\omega\sin\eta\sin\zeta,$$

$$\therefore\ G\zeta - G\epsilon = 2G\eta - c^2\sin\omega\sin\eta(\sin\zeta + \sin\eta).$$

Let η be constant; therefore

$$dF\zeta = dF\omega = dF\text{ɛ}.$$

Hence

$$G\zeta dF\zeta - G\text{ɛ}dF\text{ɛ} = 2G\eta dF\omega - c^2 \sin\omega \sin\eta \,.\, \frac{2\sin\omega\cos\eta\Delta\eta}{1 - c^2\sin^2\eta\sin^2\omega}:$$

$$\text{or}\quad \tfrac{1}{2}V\zeta - \tfrac{1}{2}V\text{ɛ} = G\eta.F\omega - \frac{\Delta\eta}{\tan\eta}.\int_0 \frac{c^2\sin^2\eta\sin^2\omega}{1 - c^2\sin^2\eta\sin^2\omega}dF\omega.$$

But the last term

$$= \surd{-\mathrm{T}}\,\{\Pi(-c^2\sin^2\eta, c, \omega) - F\},$$

and, by equation (30),

$$G\eta = \surd{-\mathrm{T}}\left\{\frac{\Pi_c(-c^2\sin^2\eta)}{F_c} - 1\right\};$$

which indeed might be here at once inferred by making $\omega = \frac{1}{2}\pi$. Hence

$$\surd{-\mathrm{T}}\mathrm{P}(-c^2\sin^2\eta, c, \omega) = \tfrac{1}{2}V(c\,\text{ɛ}) - \tfrac{1}{2}V(c\,\zeta)\ldots(48),$$

which throws a new light on equation (34).

The simpler integral $V(c\,\omega)$ now claims a full examination.

XV. Since

$$G(n\pi + \omega) = G\omega, \text{ and } F(n\pi + \omega) = F(n\pi) + F\omega,$$

therefore

$$V(n\pi + \omega) = \int G(n\pi + \omega).dF(n\pi + \omega) = \int G\omega.dF\omega = V(n\pi) + V\omega.$$

Also, since F, E, G are *odd* functions of ω, Υ and V are *even* functions; therefore

$$V(n\pi \pm \omega) = V(n\pi) + V\omega.$$

Let $n = 1$, therefore

$$V(\pi - \omega) = V(\pi + \omega) = V\pi + V\omega.$$

If then $\omega = \pi$, we get

$$0 = V(2\pi) = 2V\pi;\ \text{or}\ V\pi = 0 \ldots\ldots\ldots\ldots(49).$$

And since generally $V\{(n+1)\pi\} = V(n\pi) + V\pi$, we prove in succession that $V(2\pi) = 0$, $V(3\pi) = 0$, &c., and generally $V(n\pi) = 0$, when n is integer. Hence

$$V(n\pi \pm \omega) = V\omega \ldots\ldots\ldots\ldots\ldots\ldots(50).$$

This property suits V for tabulation, much better than Υ.

If ω begins from 0, and increases to π, V increases while G is positive; that is, up to $\omega = \frac{1}{2}\pi$, where G becomes $G_c = 0$. After this G becomes negative; indeed $G(\pi - \omega) = -G\omega$; so that V decreases after $\omega = \frac{1}{2}\pi$. Consequently, the maximum value of V is at $\omega = \frac{1}{2}\pi$, or when $V = V_c$.

Again, since

$$G\omega + G\omega^\circ = c^2 \sin\omega \sin\omega^\circ = \frac{c^2 \sin\omega \cos\omega}{\Delta\omega},$$

and $$dF\omega = -dF\omega^\circ = \frac{d\omega}{\Delta\omega};$$

multiply these together; therefore

$$dV\omega - dV\omega^\circ = \frac{c^2 \sin\omega \cos\omega d\omega}{1 - c^2 \sin^2\omega};$$

or $$V\omega - V\omega^\circ = \text{const.} - \tfrac{1}{2}\log(1 - c^2 \sin^2\omega).$$

Let $\omega = 0$, $\omega^\circ = \frac{1}{2}\pi$; therefore const. $= -V_c$, or

$$V\omega^\circ - V\omega = V_c + \log \Delta\omega \text{ .... (51).}$$

Cor. Let $\omega = \frac{1}{2}\pi$, $\omega^\circ = 0$; therefore

$$-2V_c = \log b, \quad V_c = \tfrac{1}{2}\log b^{-1} \text{ .... (51}a\text{).}$$

Thus, if b be infinitesimal, V_c is infinite. Nevertheless, even for small values of b, V_c is of very moderate amount, since it is only a logarithm.

When c is infinitesimal, $G(c\,\omega)$ vanishes for all values of ω; hence so also does $V(c\,\omega)$.

So if, $F\psi = 2F\omega$, or $\sin\psi = \dfrac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega}$,

and $G\psi = 2G\omega - c^2 \sin^2\omega \sin\psi$, we get

$$V\psi = \int_0 G\psi . dF\psi = \int_0 \{2G\omega - c^2 \sin^2\omega \sin\psi\}\, 2dF\omega,$$

or $$V\psi - 4V\omega = \int_0 -2c^2\sin^2\omega \,.\, \frac{2\sin\omega\cos\omega\Delta\omega}{1 - c^2\sin^4\omega} \,.\, \frac{d\omega}{\Delta\omega}$$

$$= \log(1 - c^2\sin^4\omega)\text{. . . . .(52).}$$

Equations (51), (51*a*), (52) are in close analogy with those established by Legendre concerning the function Υ, the integrations being the very same.

XVI. We proceed to apply Lagrange's scale to V.

Since $CG = C_1G_1 + C_1c_1 \sin\omega_1$,

or $$CG - C_1G_1 = \tfrac{1}{2}Cc^2 \frac{\sin\omega\cos\omega}{\Delta(c\omega)},$$

also $$\frac{dF}{C} = \tfrac{1}{2}\frac{dF_1}{C_1} = \frac{d\omega}{C\Delta(c\omega)};$$

multiply the two equations; therefore

$$dV - \tfrac{1}{2}dV_1 = \tfrac{1}{2}c^2 \frac{\sin\omega\cos\omega d\omega}{1 - c^2\sin^2\omega};$$

or $$V - \tfrac{1}{2}V_1 = -\tfrac{1}{2}\log\Delta(c\omega)\text{. . . . .(53),}$$

which, for a logarithmic Π, replaces (42) for the circular Π.

$$\left.\begin{array}{ll}\text{As we have} & C\{G(c\omega)-G(c\omega^\circ)\}=2C_1G(c_1\omega_1)\\ \text{so also} & V(c\omega)+V(c\omega^\circ)=V(c_1\omega_1)+V_c\end{array}\right\}\ \ldots\ldots(54).$$

If we repeat (53) n times, we find

$$V-2^{-n}V_n=-2^{-1}\log\Delta-2^{-2}\log\Delta_1-2^{-3}\log\Delta_2-\text{\&c. to } n \text{ terms.}$$

Also, since $c_n=0$, when $n=\infty$, so is $V_n=0$; therefore

$$V=-\tfrac{1}{2}\log\Delta-\tfrac{1}{4}\log\Delta_1-\tfrac{1}{8}\log\Delta_2-\text{\&c.}\ \ldots\ldots(55),$$

which is analogous to the development (44). In fact, applying the last to (48), and observing that to form $\eta_1\eta_2\eta_3\ldots$ from η, and $\omega_1, \omega_2, \omega_3\ldots$ from ω, by Lagrange's scale, and then to form $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by coupling ω and η, ω_1 and η_1, &c. amounts to forming $\zeta, \zeta_1, \zeta_2, \zeta_3\ldots$ by Lagrange's scale; and similarly of $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3\ldots$, we get

$$\sqrt{-T}P(-c^2\sin^2\eta, c, \omega)=\tfrac{1}{2}V(c\varepsilon)-\tfrac{1}{2}V(c\zeta)$$

$$=\tfrac{1}{4}\log\frac{\Delta\zeta}{\Delta\varepsilon}+\tfrac{1}{8}\log\frac{\Delta\zeta_1}{\Delta\varepsilon_1}+\tfrac{1}{16}\log\frac{\Delta\zeta_2}{\Delta\varepsilon_2}+\text{\&c.}$$

$$=-\tfrac{1}{4}\log\frac{1+D_1}{1-D_1}-\tfrac{1}{8}\log\frac{1+D_2}{1-D_2}-\tfrac{1}{16}\log\frac{1+D_3}{1-D_3}-\text{\&c.}\ldots(56),$$

if $D=c\sin\eta\sin\omega$. This equation is the transformation of (44) to the case of a logarithmic P, and apparently must be actually used to approximate to $\sqrt{-T}P$, until tables of V are calculated.

When c is very near to 1, we may invert the method. Give to (53) the form

$$V=\log\Delta'+2V',$$

or $$(V+\log\Delta)=(\log\Delta-\log\Delta')+2(V'+\log\Delta').$$

Repeating this n times, we get

$$V+\log\Delta=\log\frac{\Delta}{\Delta'}+2\log\frac{\Delta'}{\Delta''}+\ldots+2^{n-1}\log\frac{\Delta^{(n-1)}}{\Delta^{(n)}}+2^n(V^{(n)}+\log\Delta^{(n)}).$$

Now $2^nG^{(n)}=2^nE^{(n)}-2^n.\dfrac{F^{(n)}}{C^{(n)}}.E_c^{(n)}$. Also $\dfrac{F}{C}=2^n.\dfrac{F^{(n)}}{C^{(n)}}$,

$$\therefore\ 2^nG^{(n)}=2^n.E^{(n)}-\frac{F}{C}.E_c^{(n)}.$$

Again, when b is very small,

$$E=\sin\omega+\tfrac{1}{2}b^2\textstyle\int_0\tan^2\omega d\sin\omega,$$

whence $$2^n.E^{(n)}=2^n.\sin\omega^{(n)}+2^{n-1}b^{(n)2}.\textstyle\int_0\tan^2\omega^{(n)}d\sin\omega^{(n)}.$$

Now if $\omega < \frac{1}{2}\pi$, the series $\omega, \omega', \omega'', \omega'''\ldots$ decrease towards a limit $\dot{\omega} < \frac{1}{2}\pi$, so that the last integral is finite and independent of n; while $2^n b^{(n)2}$ is infinitesimal when n is infinite. Hence, for infinite values of n, $2^n E^{(n)} = 2^n . \sin\dot{\omega}$.

It is still easier to see that $\Delta^{(n)} = \cos\dot{\omega}$, when $n = \infty$.

Also $E_e^{(n)} = 1$. Hence

$$2^n . G^{(n)} = 2^n . \sin\dot{\omega} - \frac{F}{C}.$$

But $\dfrac{F^{(n)}}{B^{(n)}} = \dfrac{F}{B}$, and $B^{(n)}$ converges to $\frac{1}{2}\pi$, $F^{(n)}$ to $\dfrac{d\dot{\omega}}{\cos\dot{\omega}}$.

$$\therefore\ 2^n . V^{(n)} \text{ or } 2^n . \int_0 G^{(n)} dF^{(n)} = 2^n . \int_0 \frac{\sin\dot{\omega}}{\cos\dot{\omega}}\, d\dot{\omega} - \tfrac{1}{2}\pi \int_0 \frac{FdF}{BC},$$

$$\left.\begin{aligned} &2^n \{ V^{(n)} + \log\cos\dot{\omega} \} = -\tfrac{1}{4}\pi . \frac{F^2}{BC} \\ \text{or}\quad &2^n \{ V^{(n)} + \log\Delta^{(n)} \} = -\tfrac{1}{4}\pi . \frac{F^2}{BC} \end{aligned}\right\} \text{ when } n = \infty \ldots(57).$$

Finally, then,

$$V = -\tfrac{1}{4}\pi . \frac{F^2}{BC} + \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2 \log\frac{\Delta''}{\Delta'''} + \&c. \ldots(58),$$

which converges excellently when c is near to 1.

XVII. As we had a new integral H so related to G that $H - G = \frac{1}{2}\pi . \dfrac{F}{BC}$, it is well to conceive of W similarly related to V. Namely, as $V = \int_0 GdF$, so let $W = \int_0 HdF$;

$$\therefore\ W - V = \tfrac{1}{4}\pi\, \frac{F^2}{BC} \ldots\ldots\ldots\ldots(59).$$

This indicates that the last series is a development of W, analogous to (12),

$$W = \log\frac{1}{\Delta'} + 2\log\frac{\Delta'}{\Delta''} + 2^2 \log\frac{\Delta''}{\Delta'''} + \&c. \ldots\ldots(60).$$

Again, if $\sin\eta = \sqrt{-1}\tan\theta$, we obtain from $F(c\eta) = \sqrt{-1}\, F(b\theta)$, and from (13),

$$\left.\begin{aligned} V(c\eta) &= W(b\theta) + \log\cos\theta \\ W(c\eta) &= V(b\theta) + \log\cos\theta \end{aligned}\right\} \ldots\ldots\ldots\ldots(61),$$

in which it is remarkable that $\sqrt{-1}$ has vanished entirely. In truth, the transformation $\sin\eta = \sqrt{-1}\tan\theta$ is nothing but a device for enabling the sines and cosines of Trigonometry to do duty for *hyperbolic* sines and cosines; and we might in all cases evade $\sqrt{-1}$ in this transformation by having recourse to hyperbolic functions. If capital letters denote these, thus,

$$\text{Cos}\,x = \tfrac{1}{2}(e^x + e^{-x}),\ \text{Sin}\,x = \tfrac{1}{2}(e^x - e^{-x}),\ \&c.,$$

so that $\text{Cos}^2 x - \text{Sin}^2 x = 1$, and $1 - \text{Tan}^2 x = \text{Sec}^2 x$;

it is manifest that, by assuming $\sin\omega = \text{Tan}\,x$, we get

$$\cos\omega = \text{Sec}\,x,\ \sec\omega = \text{Cos}\,x,\ \tan\omega = \text{Sin}\,x,$$

$$d\omega = \text{Sec}\,x dx,\ dx = \sec\omega d\omega.$$

Hence $F(c\omega) = \int_0 \frac{\sec\omega d\omega}{\sqrt{(1+b^2\tan^2\omega)}} = \int_0 \frac{dx}{\sqrt{(1+b^2\text{Sin}^2 x)}} = \phi(b, x).$

Thus, by adopting the double form F and ϕ, we might deal with *real* functions only; and equations (61) seem to indicate that the integrals of the second order, V and W, recover common sines and cosines, which were displaced by hyperbolic sines and cosines in the integrals of the first order.

The equations (61) facilitate many transformations.

We may moreover give another form to (58) by slightly modifying the equation of reduction.

Since $V = \log\Delta' + 2V'$, $V + \log\cos\omega = \log(\Delta'\cos\omega) + 2V'$ $= \log(\Delta'\cos\omega\sec^2\omega') + 2(V' + \log\cos\omega')$. But $\Delta'.\cos\omega\sec^2\omega'$ $= 1 - b'\tan^2\omega'$; therefore

$$V + \log\cos\omega = \log(1 - b'\tan^2\omega') + 2(V' + \log\cos\omega')\ldots(62).$$

Repeat this n times; observe that $2^n(V^{(n)} + \log\cos\omega^{(n)})$ converges to $-\tfrac{1}{4}\pi.\frac{F^2}{BC}$; therefore

$$W + \log\cos\omega = \log(1 - b'\tan^2\omega') + 2\log(1 - b''\tan^2\omega'')$$
$$+ 2^2\log(1 - b'''\tan^2\omega''') + \&c.$$

Change $\sin\omega$ into $\sqrt{-1}\tan\theta$, $\sin\omega'$ into $\sqrt{-1}\tan\theta'$, &c. Observe that, by (61), we get

$$W(c\omega) = V(b\theta) + \log\cos\theta = V(b\theta) - \log\cos\omega;$$

hence $V(b\theta) = \log(1 + b'\sin^2\theta') + 2\log(1 + b''\sin^2\theta'') + \&c.,$

where $\theta, \theta', \theta''\ldots$ follow the scale of Gauss,

$$\sin\theta = \frac{(1 + b')\sin\theta'}{1 + b'\sin^2\theta'},\ \text{or}\ 1 + b'\sin^2\theta' = \frac{c'}{\sqrt{c}}.\frac{\sin\theta'}{\sin\theta};$$

or $$V(b\theta) = \log\left(\frac{c'}{\sqrt{c}}.\frac{\sin\theta'}{\sin\theta}\right) + 2\log\left(\frac{c''}{\sqrt{c'}}.\frac{\sin\theta''}{\sin\theta'}\right) + \&c.$$

Let $\theta = \frac{1}{2}\pi$, therefore $\theta' = \frac{1}{2}\pi = \theta'' = \theta''' = \&c....$, therefore

$$V_b = \log \frac{c'}{\sqrt{c}} + 2 \log \frac{c''}{\sqrt{c'}} + 2^2 \log \frac{c'''}{\sqrt{c''}} + \&c.... = -\log \sqrt{c},$$

and $V(b\theta) = V_b + \log \dfrac{\sin\theta'}{\sin\theta} + 2\log \dfrac{\sin\theta''}{\sin\theta'} + 2^2\log \dfrac{\sin\theta'''}{\sin\theta''} + \&c....(63)$.

This is adapted to the case of b very small, and b is here the modulus: hence the new series has no real advantage; for it is less convenient than (55). Yet we may step back to $W(c\omega)$, and write

$$1 - b' \tan^2\omega' = \frac{c'}{\sqrt{c}} \cdot \frac{\tan\omega'}{\tan\omega};$$

therefore $W(c\omega) + \log\cos\omega$

$$= V_b + \log \frac{\tan\omega'}{\tan\omega} + 2 \log \frac{\tan\omega''}{\tan\omega'} + 2^2 \log \frac{\tan\omega'''}{\tan\omega''} + \&c....(64),$$

which is adapted to the case of c near to 1.

For $\log\cos\omega - V_b$ we may write $\log(\sqrt{c}\cos\omega)$.

A new development of G by the scale of Gauss, bearing analogy to (63), may deserve notice. Since

$$V(b\theta) = \log(1 + b' \sin^2\theta') + 2\,V(b'\theta');$$

as the series itself indicates; we get, by differentiating, since

$$\frac{F(b\theta)}{B} = \frac{F(b'\theta')}{B'},$$

$$BG(b\theta) = \frac{2b' \sin\theta' \cos\theta'}{1 + b' \sin^2\theta'} . B'\Delta(b'\theta') + 2B'G(b'\theta')$$

$$= 2B'b' \sin\theta' \cos\theta + 2B'G(b'\theta')........(63a).$$

It is easy to shew that $2^n.G(b^{(n)}\theta^{(n)})$ vanishes when $n = \infty$; therefore

$$BG(b\theta) = 2B'b' \sin\theta' \cos\theta$$
$$+ 2^2B''b'' \sin\theta'' \cos\theta' + 2^3B'''b''' \sin\theta''' \cos\theta'' + \&c....(63b).$$

But this is less simple than (8).

XVIII. To approximate to $V(c\omega)$ is now as easy as to find $G(c\omega)$ or $E(c\omega)$, except only that we have no tables of it ready calculated. But it is not without interest to consider the result of differentiating V with reference to c as the variable; for which purpose we step back to F, E, and G.

Let $a = \dfrac{B}{C}$, $x = \frac{1}{2}\pi . \dfrac{F(c\omega)}{C}$. Then, in the scale whose index is n, to pass from $c\omega$ to new elements $c_1\omega_1$ changes ax to na,* nx. Moreover, in the higher theory, if we adopt, with Dr. Gudermann, the notation of *hyperbolic* sines and cosines, the development of G admits of the form

$$CG(c\omega) = \pi\left\{\frac{\sin 2x}{\text{Sin}\,\pi a} + \frac{\sin 4x}{\text{Sin}\,2\pi a} + \frac{\sin 6x}{\text{Sin}\,3\pi a} + \&\text{c.}...\right\},$$

which things suggest the advantage of making a rather than c the base of variation.

The same process which demonstrates $\frac{1}{2}\pi = F_cE_b + F_bE_c - F_bF_c$ shews, in passing, that $\frac{1}{2}\pi.\frac{dc}{da} = -b^2c.F_c^2$. In the common treatises we have

$$\frac{dF}{dc} = \frac{E}{b^2c} - \frac{F}{c} - \frac{c\sin\omega\cos\omega}{b^2\Delta\omega}, \text{ when } \omega \text{ is constant;}$$

$$\therefore\ \tfrac{1}{2}\pi.\frac{dF}{da} = -F_c^2\{E - b^2F - c^2\sin\omega\sin\omega^\circ\}.....(65).$$

Hence $\quad \frac{1}{2}\pi.\frac{dF_c}{da} = -F_c^2\{E_c - b^2F_c\}$, when $\omega = \frac{1}{2}\pi$.

Multiply the last by $\frac{F}{F_c}$ and subtract from (65), therefore

$$\tfrac{1}{2}\pi\left\{\frac{dF}{da} - \frac{F}{F_c}.\frac{dF_c}{da}\right\} = -F_c^2\{G - c^2\sin\omega\sin\omega^\circ\},$$

whence $\frac{1}{2}\pi.\frac{d}{da}\left(\frac{F}{F_c}\right)$ or $\frac{dx}{da} = F_c.G(c\omega^\circ)$, when ω is const...(66).

Let A stand temporarily for CG or F_cG; *i.e.* $A = F_cE - E_cF$; and let us seek for $\frac{dA}{da}$.

$$\text{Since } \frac{dE}{dc} = -\frac{F-E}{c},\quad \tfrac{1}{2}\pi.\frac{dE}{da} = b^2F_c^2(F-E).$$

$$\text{Also } \tfrac{1}{2}\pi.\frac{dA}{da} = F_c.\tfrac{1}{2}\pi\frac{dE}{da} - F.\tfrac{1}{2}\pi.\frac{dE_c}{da} + E.\tfrac{1}{2}\pi\frac{dF_c}{da} - E_c.\tfrac{1}{2}\pi.\frac{dF}{da}$$

$$= -E_cF_c^2c^2\sin\omega\sin\omega^\circ.........................(67),$$

by mere substitutions.

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* For $a = \frac{B}{C}$, $a_1 = \frac{B_1}{C_1}$ gives $a = \frac{1}{n}a_1$ in that scale.

It is observable, that we also have

$$\tfrac{1}{2}\pi.\frac{d(cF_c)}{da} = -E_cF_c^2c;$$

so that
$$\frac{dA}{da} = \sin\omega\sin\omega^\circ.\frac{d(cF_c)}{da} \text{ .... } (67a).$$

In all these equations, (65)—(67a), we suppose ω to be constant. But in general, when c and ω both vary,

$$\frac{d(x)}{da} = \frac{dx}{d\omega}\frac{d\omega}{da} + \frac{dx}{da}; \text{ and } \frac{d(A)}{da} = \frac{dA}{d\omega}\frac{d\omega}{da} + \frac{dA}{da};$$

that is,
$$\frac{d(x)}{da} = \frac{\frac{1}{2}\pi}{F_c\Delta}.\frac{d\omega}{da} + A(\omega^\circ);$$

$$\tfrac{1}{2}\pi\frac{d(A)}{da} = \tfrac{1}{2}\pi\frac{d\omega}{da}.\left(\frac{dA}{d\omega}\right) - E_cF_c^2c^2\sin\omega\sin\omega^\circ.$$

Now let x be the principal variable instead of ω; and when a varies, let x be constant, or $\frac{d(x)}{da} = 0$; and eliminate $\frac{d\omega}{da}$ from the two last; observing that

$$\frac{dA}{d\omega} = F_c\Delta - \frac{E_c}{\Delta} \text{ and } A(\omega^\circ) = F_cc^2\sin\omega\sin\omega^\circ - A(\omega);$$

$$\therefore \tfrac{1}{2}\pi.\frac{dA}{da} = F_c.\Delta.A\frac{dA}{d\omega} - F_c^2c^2\sin\omega\cos\omega\Delta \text{ ......} (68).$$

Multiply by $\frac{dx}{\frac{1}{2}\pi} = \frac{d\omega}{F_c\Delta}$; and since x is constant in $\frac{d}{da}$, we may integrate for x under the d; or

$$\int_0 \frac{dA}{da}dx = \frac{d}{da}\int_0 Adx = \frac{d}{da}\int_0 GdF.\tfrac{1}{2}\pi;$$

$$\therefore \tfrac{1}{2}\pi.\frac{d}{da}V(c\omega) = \tfrac{1}{2}A^2(c\omega) - \tfrac{1}{2}F_c^2c^2\sin^2\omega \text{ ...... } (69),$$

when x is constant.

If this have no other interest, it at least shews that A^2 can be expressed in series of the cosines of $2x$ and of its multiples: for the developments of $V(c\omega)$ and $F_c^2c^2\sin^2\omega$ are known.

XIX. The same notation facilitates the management of E, G, and V in the higher scales. To avoid confusion, in the scale whose index is n, let $h\psi$ be the new elements of F

which are called $c_1\omega_1$ in the common scale. Then, since x, a change to nx, na, when $c\omega$ change to $h\psi$, we have, as *total* variations,

$$\left.\begin{aligned} \frac{dx}{da} &= \frac{\frac{1}{2}\pi}{F_c\Delta(c\omega)}\cdot\frac{d\omega}{da} + F_cG(c\omega^\circ) \\ \frac{d.nx}{nda} &= \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda} + F_hG(h\psi^\circ) \end{aligned}\right\}.$$

The left-hand member is the same in both equations, and we may at pleasure assume any *one* of the variables as constant. If it be ω, $\frac{d\omega}{da} = 0$, therefore

$$F_cG(c\omega^\circ) = F_hG(h\psi^\circ) + \frac{\frac{1}{2}\pi}{F_h\Delta(h\psi)}\cdot\frac{d\psi}{nda} \quad \ldots.(70).$$

This is a generalization of $CG = C_1G_1 + C_1c_1 \sin\omega_1$, and supersedes a much more complicated one connecting $E(c\omega)$ with $E(h\psi)$ in Legendre.

Multiply by

$$\frac{dF(c\omega^\circ)}{F_c} = \frac{dF(h\psi^\circ)}{nF_h} = -\frac{1}{nF_h}\cdot\frac{d\psi}{\Delta(h\psi)};$$

$$\therefore\ V(c\omega^\circ) = \frac{1}{n}V(h\psi^\circ) - \frac{\frac{1}{2}\pi}{n^2F_h^2}\cdot\int_0\left(\frac{d\psi}{da}\right)\frac{d\psi}{1-h^2\sin^2\psi} \ldots(71),$$

which also is a generalization of (53).

Since Jacobi's equations shew $\frac{d\psi}{da}$ to be a rational trigonometrical function, the integral is of a lower order than F.

In Legendre's own scale, the index of which is 3, equation (70) takes the form

$$F_cG(c\omega) = F_hG(h\psi) + \tfrac{2}{3}F_c.\mu c^2\sin^2\alpha_2.\ \frac{\sin\omega\cos\omega\Delta(c\omega)}{1-c^2\sin^2\alpha_2\sin^2\omega} \ldots(70a),$$

where $F(c\alpha_2) = \frac{2}{3}F_c$, and $\tan\frac{1}{2}(\psi-\omega) = \frac{1-\mu}{\mu}\tan\omega$.

Multiply by $\frac{dF(c\omega)}{F_c} = \frac{dF(h\psi)}{3F_h}$, and in place of (71) we have

$$V(c\omega) = \tfrac{1}{3}V(h\psi) - \tfrac{1}{3}\mu\log(1-c^2\sin^2\alpha_2\sin^2\omega)\ldots(71a).$$

This easily gives a new development of V, converging far more rapidly than (53): but until we have tables for the trisection of F_c, the trouble of calculating the constants makes this scale practically useless.

It is proper also to notice here the relation borne by V and W to Jacobi's new functions. Let q be a small fraction such that $\log q^{-1} = \pi a$, and $\Theta\Lambda$ functions such that

$$\left.\begin{aligned}\Theta &= 1 - 2q\cos 2x + 2q^{2.2}\cos 4x - 2q^{3.3}\cos 6x + \&c.\ldots\\ \Lambda &= \quad 2q^{\frac{1}{4}}\sin x - 2q^{\frac{3.3}{4}}\sin 3x + 2q^{\frac{5.5}{4}}\sin 5x - \&c.\ldots\end{aligned}\right\};$$

then, among other equations, Jacobi has proved that

$$\sqrt{(1 - c^2\sin^2\omega)} = \sqrt{b}.\,\frac{\Theta(q,\, \tfrac{1}{2}\pi + x)}{\Theta(qx)},$$

$$\text{and } \cos\omega = \sqrt{\frac{b}{c}}\,.\,\frac{\Lambda(q,\, x + \tfrac{1}{2}\pi)}{\Theta(qx)}.$$

Legendre has demonstrated (2nd Supplement, § XI.) that

$$\frac{2F_c}{\pi}\,G(c\omega) = \frac{\Theta'(qx)}{\Theta(qx)}.$$

If we multiply by $\tfrac{1}{2}\pi.\dfrac{dF(c\omega)}{F_c} = dx$, and integrate, we get

$$V = \log(\beta\Theta),$$

where β is a function of c.

When $\omega = \pi$, $V = 0$; therefore $\beta\Theta = 1$, or

$$\beta^{-1} = \Theta(q,\, \pi) = 1 - 2q + 2q^{2.2} - 2q^{3.3} + \&c.\ldots,$$

a series which is known to be equal to $\sqrt{\dfrac{2bF_c}{\pi}}$.

But it suffices to write $V = \log\dfrac{\Theta(q,\, x)}{\Theta(q,\, \pi)}$ ....(72).

This elegant relation is obscurely expressed by Legendre under the following form

$$\log\Theta(qx) = \Upsilon(c\omega) - \tfrac{1}{2}\,\frac{E_c}{F_c}\,.F^2(c\omega) + \tfrac{1}{2}\log\frac{2bF_c}{\pi}.$$

Had he used the integral V, he would not have overlooked the following curious inference, which would certainly have had a charm for him.

Since (by the form of Θ when resolved into factors)

$$\Theta(q^n,\, nx) = \text{const.} \times \Theta(q,\, x).\Theta\left(q,\, x + \frac{\pi}{n}\right).\Theta\left(q,\, x + \frac{2\pi}{n}\right)\ldots\ldots$$
$$\Theta\left(q,\, x + \frac{n-1}{n}\,\pi\right):$$

differentiate logarithmically, observing that

$$d \log \Theta = dV = GdF,$$

and putting h the same function of nx as c is of x. Therefore

$$nF_h . G(q^n, nx)$$

$$= F_c\left\{G(q, x) + G\left(q, x + \frac{\pi}{n}\right) + \ldots + G\left(q, x + \frac{n-1}{n}\pi\right)\right\} \ldots (73),$$

in which we write after G the elements qx instead of $c\omega$, but meaning the same quantity.

A process which is laborious in Legendre becomes easy by aid of (72), (59), (61).

When $\sin\omega = \sqrt{-1}\tan\theta$, and $F(c\omega) = \sqrt{-1}.F(b\theta)$,

$$x = \tfrac{1}{2}\pi . \frac{F(c\omega)}{C} = \sqrt{-1} . \frac{B}{C} . \frac{F(b\theta)}{B} = \sqrt{-1}.at.$$

Call $T(qt)$ the value assumed by $\Theta(q, x)$; that is,

$$\Theta(qx) = 1 - 2q \operatorname{Cos} 2at + 2q^{2.2} \operatorname{Cos} 4at - \&c. \ldots = T(q, t),$$

$$\therefore \log \beta T(qt) = V(c\omega) = W(b\theta) + \log\cos\theta, \text{ by (61)},$$

$$= V(b\theta) + \tfrac{1}{4}\pi . \frac{F^2(b\theta)}{BC} + \log\cos\theta, \text{ by (59)},$$

$$= \log\gamma\Theta(rt) + \tfrac{1}{4}\pi a t^2 + \log\cos\theta,$$

if q, β become r, γ when c changes to b. Giving to β, γ their values, we get

$$\log T(qt) = \log\Theta(rt) + \tfrac{1}{4}\pi a t^2 + \log\cos\theta + \tfrac{1}{2}\log\frac{b}{ca} \ldots (74),$$

which is one of Legendre's equations, and, by means of $\cos\omega = \sqrt{\frac{b}{c}} . \frac{\Lambda(q, x + \frac{1}{2}\pi)}{\Theta(qx)}$, readily gives his beautiful result

$$\left.\begin{aligned} \sqrt{a}.T(qt) &= q^{-\left(\frac{t}{\pi}\right)^2}.\Lambda(r, t + \tfrac{1}{2}\pi) \\ \sqrt{a^{-1}}.T(rx) &= r^{-\left(\frac{x}{\pi}\right)^2}.\Lambda(q, x + \tfrac{1}{2}\pi) \end{aligned}\right\} \ldots\ldots\ldots (75);$$

so that Λ is found from T, or T from Λ, according as one or the other converges best.

In this part of the subject it must be added, that the arc Ω, elicited by Legendre from Jacobi's theory, is no other quantity than that which I have called $\sqrt{TP}$; so that, according to the meaning of $\overset{s}{\Omega}$ in equation (26),

$$\tan\overset{s}{\Omega}=\frac{2q\sin 2x.\text{Sin}\,2at-2q^{2.2}\sin 4x.\text{Sin}\,4at+2q^{3.3}\sin 6x\,\text{Sin}\,6at-\&c.}{1-2q\cos 2x.\text{Cos}\,2at+2q^{2.2}\cos 4x.\text{Cos}\,4at-\&c....}\,.$$

But it suffices to mention the fact. Dr. Gudermann has several other elegant approximations to $\overset{s}{\Omega}$ by this higher theory.

I have thought that we might call the function $\sqrt{\pm}\,T\{\Pi-F\}$ the *Principal Compound* of the Third Elliptic Species, and $\sqrt{\pm}\,TP$ its *Fluctuant:* also G the Fluctuant of E, H its Companion; Υ the First Integral of the Second Order, V its Fluctuant, and W the Companion of V. To avoid a perpetual appropriation of capital letters, some such notation as flE for G, $fl\Pi$ for P, &c. may sometimes be advisable.

XX. Lastly, it may be worth while to touch on a neglected side of the subject; but, as it involves no difficulty of principle, and possibly is more curious than useful, I may be brief.

In fixing the standard forms of F, E, Π, two arbitrary limitations are introduced,—to use circular sines, and not hyperbolic; and, to make the multiplier a negative proper fraction $(-c^2)$. Imaginary Amplitudes overthrow the former limitation, imaginary Moduli the other. But to change $-c^2$ into $+c^2$ involves nothing imaginary, nor indeed, within certain limits, to change b^2 into b^{-2}, which makes $c^2>1$.

For a moment put $z=\tan\omega$, and let $\psi(bz)$, $\chi(bz)$ denote what $F(c\omega)$, $E(c\omega)$ become; or

$$\psi(bz)=\int_0\frac{dz}{\sqrt{(1+z^2)}\sqrt{(1+b^2z^2)}},\quad \chi(bz)=\int_0\sqrt{\left(\frac{1+b^2z^2}{1+z^2}\right)}\cdot\frac{dz}{1+z^2}\,.$$

Mere inspection of these shews that

$$d\psi(b^{-1},z^{-1})=-\,b.\psi(bz)\text{ and }d\chi(b^{-1},z^{-1})=-\,b^{-1}.\chi(bz).$$

To change z into z^{-1} changes ω into $\frac{1}{2}\pi-\omega$; and if $c=\sin\gamma$, to change b into b^{-1}, or $\cos\gamma$ into $\sec\gamma$, is equivalent to changing $\sin\gamma$ into $\sqrt{-1}\tan\gamma$. Hence, integrating the two last, we find, if $\omega+\theta=\frac{1}{2}\pi$,

$$\left.\begin{aligned}b^{-1}F(\sqrt{-1}\tan\gamma,\theta)+F(\sin\gamma,\omega)&=F_c\\ bE(\sqrt{-1}\tan\gamma,\theta)+E(\sin\gamma,\omega)&=E_c\end{aligned}\right\}\text{......(76)}.$$

It is easy then to transform the developments which express F_cF_b and E_cE_b in terms of c^2, into others in which $-c^2b^{-2}$ stands for $+c^2$. For

$$b^{-1}F(\sqrt{-1}\tan\gamma,\tfrac{1}{2}\pi)=F_c\text{ and }bE(\sqrt{-1}\tan\gamma,\tfrac{1}{2}\pi)=E_c\text{...(76}a\text{)}.$$

It farther follows that

$$bG(\sqrt{-1}\tan\gamma, \theta) = b.E(\sqrt{-1}\tan\gamma, \theta) - b^{-1}\frac{E_c}{F_c}F(\sqrt{-1}\tan\gamma, \theta)$$

$$= \{E_c - E(\sin\gamma, \omega)\} - \frac{E_c}{F_c}\{F_c - F(\sin\gamma, \omega)\}$$

$$= - G(\sin\gamma, \omega) \dots\dots\dots\dots (77).$$

Hence also, if A stands for $F_cE - E_cF$, we obtain

$$A(\sqrt{-1}\tan\gamma, \theta) = - A(\sin\gamma, \omega) \dots\dots\dots (78).$$

Moreover, since $V = \int_0 GdF$,

$$V(\sqrt{-1}\tan\gamma, \theta) = \int - b^{-1}G(c\omega) \times - bdF(c\omega) = \int dV(c\omega)$$

$$= V(c\omega) - V_c \dots\dots\dots\dots (79).$$

When $\omega = 0$, $\quad V(\sqrt{-1}\tan\gamma, \tfrac{1}{2}\pi) = - V_c$;

which is evidently correct: for since $V_c = \frac{1}{2}\log b^{-1}$, it does but change the sign, if we commute b with b^{-1}.

For (79) we may thus also write

$$V(\sqrt{-1}\tan\gamma, \theta) = V(\sin\gamma, \omega) + \tfrac{1}{2}\log\cos\gamma \dots (79^*).$$

It might at first seem possible to reduce the circular Π to integrals V which have imaginary moduli; but these equations shew that such integrals fall back into the common form. Moreover, if in Π the modulus is imaginary, the integral is logarithmic exactly when by its parameter it might have been judged to be circular. To exhibit this may well close our subject.

When c changes to $\sqrt{-1}\tan\gamma$, our ordinary logarithmic parameter $-c^2\sin^2\eta$ becomes $\tan^2\gamma.\sin^2\eta$, which is positive and apparently circular. Observe that if $c = \sin\gamma$,

$$\sqrt{(1 + \tan^2\gamma\sin^2\eta)}\cot\eta = b^{-1}\sqrt{(1 - c^2\cos^2\eta)}\cot\eta,$$

$$\text{and} \quad \frac{dF(\sqrt{-1}\tan\gamma, \theta)}{1 + \tan^2\gamma\sin^2\eta\sin^2\theta} = \frac{-b^3.dF(c\omega)}{b^2 + c^2\sin^2\eta\cos^2\omega},$$

the denominator of which $=(1-c^2\cos^2\eta)-c^2\sin^2\eta\sin^2\omega$. Hence if we change η all through into $\frac{1}{2}\pi - \eta$, we have

$$\sqrt{-T}\Pi(\tan^2\gamma\cos^2\eta, \sqrt{-1}\tan\gamma, \theta) = -b^2\frac{\Delta(c\eta)}{\cot\eta}.\int\frac{dF(c\omega)}{\Delta^2(c\eta)-c^2\cos^2\eta\sin^2\omega}$$

$$= \frac{-b^2\tan\eta}{\Delta(c\eta)}.\int\frac{dF(c\omega)}{1-c^2\sin^2\eta^\circ.\sin^2\omega} = -\frac{\Delta(c\eta^\circ)}{\tan\eta^\circ}\Pi(-c^2\sin^2\eta^\circ, c, \omega) + \text{const.}$$

Observe that $\frac{\Delta(c\eta^\circ)}{\tan\eta^\circ} = \sqrt{-T}(-c^2\sin^2\eta^\circ)$; then we obtain

$$\sqrt{-T}\Pi(\tan^2\gamma\cos^2\eta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \omega)$$
$$= \sqrt{-T}\Pi(-\sin^2\gamma\sin^2\eta^\circ, \sin\gamma, \tfrac{1}{2}\pi)$$
$$= \sqrt{-T}\Pi_c(-c^2\sin^2\eta^\circ) \dots\dots (80),$$

where the integrals are all logarithmic, and $\sin\gamma = c$, connecting η and η°.

To transform them into circulars, let

$$\sin\eta^\circ = \sqrt{-1}.\tan\delta_\circ,\ \cos^2\eta = \frac{b^2\sin^2\eta^\circ}{\Delta^2(c\eta^\circ)},$$

$$\text{or } \tan^2\gamma\cos^2\eta = \frac{c^2\sin^2\eta^\circ}{1-c^2\sin^2\eta^\circ} = \frac{-c^2\tan^2\delta^\circ}{1+c^2\tan^2\delta_\circ} = \frac{-\cot^2\delta}{1+\cot^2\delta} = -\cos^2\delta.$$

Hence $\sqrt{T}\Pi(-\cos^2\delta, \sqrt{-1}\tan\gamma, \theta) + \sqrt{T}\Pi(\cot^2\delta, \sin\gamma, \omega)$
$$= \sqrt{T}\Pi_c(\cot^2\delta) \dots\dots (81).$$

In all of these, from (76) to (81), we suppose $\omega + \theta = \frac{1}{2}\pi$.

ON TWO NEW METHODS OF DEFINING CURVES OF THE SECOND ORDER, TOGETHER WITH NEW PROPERTIES OF THE SAME DEDUCIBLE THEREFROM.

By Professor Steiner.*

(*Extract from a paper read before the Berlin Academy of Science, March* 1852.)

Section I.

The two following methods of generating the Conic Sections are in a measure analogous to, and indeed embrace, the two known methods of generation by means of the two foci, or one focus and the corresponding directrix. The first method consists in making the sum or difference of the lengths of two tangents from the generating point to two given fixed circles, equal to a given constant, instead of, as before, considering the sum or difference of the distances from the said point to the two foci themselves as given and constant. In the second method here employed, the simple directrix is replaced by any number of given

* Translated by Dr. T. A. Hirst.