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EDITED BY W. THOMSON, M.A.

FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE,

AND PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW.

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TO DEVELOP $(\cos x)^a$ IN A SERIES OF COSINES FOR ALL VALUES OF a.

By Francis W. Newman.

Let $X = \log(2\cos x) = \cos 2x - \frac{1}{2}\cos 4x + \frac{1}{3}\cos 6x - \&c\ldots$ when x is between $\pm\frac{1}{2}\pi$; therefore

$$(2\cos x)^a = 1 + \frac{a}{1}X + \frac{a^2}{1.2}X^2 + \frac{a^3}{1.2.3}X^3 + \&c\ldots$$

Now X^n contains nothing but terms made up of factors $(\cos 2mx)^r$, where m and r are positive integers; hence X^n is expressible as $a' + \Sigma b \cos 2nx$. Consequently, while x is between $\pm\frac{1}{2}\pi$, we may assume

$$(\cos x)^a = A + A_1\cos 2x + A_2\cos 4x + A_3\cos 6x + \&c\ldots\ldots(1),$$

where $A, A_1, A_2\ldots$ are unknown functions of a.

Put $y = (\cos x)^a$, therefore $ay\sin x + \frac{dy}{dx}\cos x = 0$. By substituting as usual for y and its differential coefficient in terms of x, and reducing to linear sines, we find that the differential equation will be satisfied, if

$$2aA = (a+2)A_1;\quad (a-2)A_1 = (a+4)A_2;$$
$$(a-4)A_2 = (a+6)A_3;\ \&c\ldots\ldots\ldots(2),$$

so that $A_1, A_2, A_3\ldots$ are known multiples of A; and we get

$$(\cos x)^a = A.\left\{1 + \frac{2a\cos 2x}{a+2} + \frac{2a.a-2.\cos 4x}{a+2.a+4} + \frac{2a.a-2.a-4.\cos 6x}{a+2.a+4.a+6} + \&c\ldots\right\}\ldots(3).$$

To find A, multiply the series by dx, and integrate from $x=0$ to $x=\frac{1}{2}\pi$, observing that

$$\int_0^{\frac{1}{2}\pi}\cos 2nx dx = 0,\quad \text{and} \int_0^{\frac{1}{2}\pi}(\cos x)^a dx$$

is found by making $\cos x = \sqrt{z}$, to be

$$= \tfrac{1}{2}F\{\tfrac{1}{2}(a+1), \tfrac{1}{2}\} = \frac{\Gamma.\frac{1}{2}(a+1).\Gamma\frac{1}{2}}{2.\Gamma.\frac{1}{2}(a+2)};\ \text{also}\ \Gamma\tfrac{1}{2} = \sqrt{\pi}.$$

If then we can assume the entire series, after the first term, to vanish, when every term vanishes, we shall get

$$\frac{\sqrt{\pi}.\Gamma.\frac{1}{2}(a+1)}{2.\Gamma.\frac{1}{2}(a+2)} = A.\tfrac{1}{2}\pi;\ \text{or}\ A = \frac{\Gamma.\frac{1}{2}(a+1)}{\sqrt{\pi}\Gamma.\frac{1}{2}(a+2)}\ldots(4).$$

To test the assumption, observe that

$$(2n + a + 2)\, A_{n+1} = -(2n - a)\, A_n;$$

so that for very large values of n, A_{n+1} and A_n are of opposite sign; and tend towards equality, whether a be positive or negative. Hence the latter part of the series tends to coincidence with

$$y = A_n (\cos 2nx - \cos(2n+2)x + \cos(2n+4)x - \&c.\ldots.).$$

Now

$$\int_0 y\,dx = \tfrac{1}{2} A_n \left(\frac{\sin 2nx}{n} - \frac{\sin(2n+2)x}{n+1} + \frac{\sin(2n+4)x}{n+2} - \&c.\right)$$

The series here is known to converge, and vanishes when $x = \frac{1}{2}\pi$: and by taking n a large finite number, we may make our series agree as nearly as we please with this; and unless A_n increases so rapidly with n as to give a finite value to the product, the problem is solved. Now when a is positive, the coefficients A, A_1, A_2, $A_3 \ldots$ diminish; so that A_n is finite or zero. If however a be negative, we must inquire farther.

$$\text{Now } A_n = (-1)^{n-1}.\, 2A \cdot \frac{a}{2+a} \cdot \frac{2-a}{4+a} \cdot \frac{4-a}{6+a} \ldots\ldots \frac{2n-2-a}{2n+a}$$

$$= (-1)^{n-1}.\, \frac{2Aa}{2n+a} \cdot \frac{2-a}{2+a} \cdot \frac{4-a}{4+a} \ldots\ldots \frac{2n-2-a}{2n-2+a}.$$

This may be computed from the properties of Γ, when n is very large. For we have, as approximate equations (*true* only when $n = \infty$),

$$\Gamma a = \frac{e^{-\gamma a}}{a} \cdot \frac{e^{a}}{1+a} \cdot \frac{e^{\frac{1}{2}a}}{1+\frac{1}{2}a} \ldots\ldots \frac{e^{n^{-1}a}}{1+n^{-1}a}:$$

$$\frac{\sin \pi a}{\pi a} = (1 - a^2)\left(1 - \frac{a^2}{4}\right) \ldots\ldots \left(1 - \frac{a^2}{n^2}\right):$$

Multiply the latter by the square of the former;

$$(\Gamma a)^2.\, \frac{\sin \pi a}{\pi a} = \frac{e^{-2\gamma a}}{a^2}\, e^{2a(1+2^{-1}+\ldots+n^{-1})}.\, \frac{1-a}{1+a} \cdot \frac{2-a}{2+a} \ldots\ldots \frac{n-a}{n+a}.$$

When n is very large $(1 + 2^{-1} + 3^{-1} + \ldots + n^{-1})$ nearly $= \gamma + \log n$, so that if we write $\frac{1}{2}a$ for a, we get more simply

$$(\Gamma \tfrac{1}{2}a)^2\, \frac{\sin(\frac{1}{2}\pi a)}{\frac{1}{2}\pi a} = \frac{4n^a}{a^2} \cdot \frac{2-a}{2+a} \cdot \frac{4-a}{4+a} \ldots\ldots \frac{2n-a}{2n+a};$$

whence $$A_n = (-1)^{n-1}.\, \frac{Aa^2\pi^{-1}n^{-a}}{2n-a} \cdot (\Gamma \tfrac{1}{2}a)^2 \sin \tfrac{1}{2}\pi a,$$

which is infinite when a is negative, and $n = \infty$.

Nevertheless let $(a+1)$ be positive, and $= b$, which is then between 0 and 1; therefore

$$A_n = (-1)^n\, 2A \cdot \frac{1-b}{1+b} \cdot \frac{3-b}{3+b} \cdot \frac{5-b}{5+b} \cdots\cdots \frac{2n-1-b}{2n-1+b},$$

from which we see that $A, A_1, A_2 \ldots\ldots A_n$ is still a decreasing series; consequently we can trust our conclusion, provided that a is either positive, or, if negative, less than 1.

Thus, provided that $(1+a)$ is positive, and x is between $\pm\frac{1}{2}\pi$; we have

$$(\cos x)^a = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma.\frac{1}{2}(a+1)}{\Gamma.\frac{1}{2}(a+2)} \cdot \left\{1 + \frac{2a\cos 2x}{2+a} - \frac{2a\,.\,2-a\,.\cos 4x}{2+a\,.\,4+a} + \frac{2a\,.\,2-a\,.\,4-a\,.\cos 6x}{2+a\,.\,4+a\,.\,6+a} - \&c.\right\}.$$

which includes the well-known development of $(\cos x)^{2n}$; the series terminating when $a = 2n$.

When $a = 2n-1$, $\Gamma.\frac{1}{2}(a+1)$ and $\Gamma.\frac{1}{2}(a+2)$ may be simplified, and we get

$$A = \frac{2}{\pi} \cdot \frac{2\,.\,4 \ldots\ldots 2n-2}{3\,.\,5 \ldots\ldots 2n-1}.$$

In particular, if $a = 1$, $n = 1$,

$$\tfrac{1}{4}\pi \cos x = \tfrac{1}{2} + \frac{\cos 2x}{1.3} - \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} - \&c.\ldots\ldots$$

When a reaches the limit -1, A becomes infinite, and we have

$$(\cos x)^{-1} = A\,(1 - 2\cos 2x + 2\cos 4x - 2\cos 6x + \&c.\ldots\ldots),$$

where the series is indeterminate. We may perhaps conclude that the development in this form is impossible, when $(-a)$ exceeds 1.

ON THE DETERMINATION OF THE MODULUS OF ELASTICITY OF A ROD OF ANY MATERIAL, BY MEANS OF ITS MUSICAL NOTE.

By Andrew Bell.

It is proposed in this paper to determine the modulus of elasticity of any material, by means of the musical note obtained from a rod of the material. The modulus being