THE CAMBRIDGE AND DUBLIN MATHEMATICAL JOURNAL.

EDITED BY W. THOMSON, M.A.

FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE,
AND PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW.

VOL. III.

(BEING VOL. VII. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

Δυῶν ὀνομάτων μορφὴ μία.

CAMBRIDGE:
MACMILLAN, BARCLAY, AND MACMILLAN;
GEORGE BELL, LONDON;
HODGES AND SMITH, DUBLIN.
1848

ON Γa, ESPECIALLY WHEN a IS NEGATIVE.

By Francis W. Newman.

The importance of bringing the function Γa, or $\int_0^\infty e^{-x} x^{a-1} dx$, within the elementary departments of Analysis, is acknowledged. In so doing, the more difficult steps are found to be the establishment of the equation

$$F(a, b) \text{ or } \int_0^1 x^{a-1} (1 - x)^{b-1}\, dx = \frac{\Gamma a \,.\, \Gamma b}{\Gamma (a + b)},$$

and the extension of Γa to the case of a negative. It is proposed here to indicate the order and mode of investigation, by which all may be effected, without involving any process at which the most cautious learner could justly hesitate.

We commence with the *Primary Properties of F*. We suppose a and b positive; then F is finite.

From the first principles of integration, it is obvious to prove

$$F(a, b) = F(b, a) \ldots\ldots\ldots\ldots\ldots (1),$$

$$F(a', b) < F(a, b), \text{ if } a' > a \ldots\ldots\ldots\ldots (2),$$

$$F(a, b) = \left(1 + \frac{b}{a}\right) F(a + 1, b) \ldots\ldots\ldots (3).$$

So far, all reasoners, from Euler downward, have proceeded alike. Observe now, that if $b \div a$ is an evanescent fraction, we have from the last, that $F(a, b) = F(a + 1, b)$,—or the limit of their ratio is 1, when $b \div a$ perpetually lessens. But (for conciseness merely) we may here dispense with the latter mode of expression.

It follows, that if r is any finite integer, and $(b \div a)$ evanescent, we have $F(a, b) = F(a + 1, b) = \ldots\ldots = F(a + r, b)$.

Also, if δ is a proper fraction, we see by (2) that $F(a + r + \delta, b)$ is between $F(a + r, b)$ and $F(a + r + 1, b)$. Put $r + \delta =$ any finite number α, not necessarily integer, and we find

$$F(a, b) = F(a + \alpha, b), \quad \text{if } \frac{b}{a} = 0 \text{ and } \alpha \text{ finite} \ldots\ldots (4).$$

We proceed to develop $F(a, b)$ in a series of factors.

Repeatedly applying equation (3), we get

$$F(a, b) = \frac{a + b}{a} \,.\, \frac{1 + a + b}{1 + a} \ldots\ldots \frac{n - 1 + a + b}{n - 1 + a} \,.\, F(n + a, b).$$

Write 1 for a, $n - 1$ for n, in the last; then

$$F(1, b) = \frac{1 + b}{1} \,.\, \frac{2 + b}{2} \ldots\ldots \frac{n - 1 + b}{n - 1}\, F(n, b).$$

Divide the former by the latter, observing that $F(1, b) = b^{-1}$;

$$\therefore F(a, b) = \frac{a+b}{ab} \cdot \frac{1 \cdot 1 + a + b}{1 + a \cdot 1 + b} \cdot \frac{2 \cdot 2 + a + b}{2 + a \cdot 2 + b} \cdot \text{ to } n \text{ factors}$$

$$\times \frac{F(n + a, b)}{F(n, b)}.$$

Now in equation (4) write n for a, and a for α; make $n = \infty$, and we find $F(n, b) = F(n + a, b)$, provided that a is finite: hence

$$F(a, b) = \frac{a+b}{ab} \cdot \frac{1 + (a+b)}{1 + a \cdot 1 + b} \cdot \frac{1 + \frac{1}{2}(a+b)}{1 + \frac{1}{2}a \cdot 1 + \frac{1}{2}b} \cdot \frac{1 + \frac{1}{3}(a+b)}{1 + \frac{1}{3}a \cdot 1 + \frac{1}{3}b} \ldots \&c. \ldots \ldots (5).$$

Examine this result, and it appears that if we venture to write

$$\psi(a) \text{ for } a(1 + a)(1 + \tfrac{1}{2}a)(1 + \tfrac{1}{3}a) \&c. \ldots\ldots$$

we shall have $F(a, b) = \psi(a + b) \div (\psi a, \psi b)$; which resolves F, a function of two variables, into ψ, a function of one. But a little inspection shews ψ to be infinite; since

$$\log \psi a = \log a + \log(1 + a) + \log(1 + \tfrac{1}{2}a) + \log(1 + \tfrac{1}{3}a) + \&c.$$

$$= \log a + (1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c.)\, a - \tfrac{1}{2} S_2 a^2 + \tfrac{1}{3} S_3 a^3 - \&c.\ldots\ldots$$

by developing, when a is < 1. Now as the coefficient of a is infinite, so is ψ. Nevertheless, that suggests another assumption, free from this inconvenience. Suppose

$$\psi a = a \cdot \frac{1 + a}{e^a} \cdot \frac{1 + \frac{1}{2}a}{e^{\frac{1}{2}a}} \cdot \frac{1 + \frac{1}{3}a}{e^{\frac{1}{3}a}} \cdot \&c. \&c. \ldots\ldots (6),$$

$$\therefore \log \psi a = \log a - \tfrac{1}{2} S_2 a^2 + \tfrac{1}{3} S_3 a^2 - \tfrac{1}{4} S_4 a^4 + \&c. \ldots\ldots$$

which is finite. And since $e^{r+s} = e^r \cdot e^s$, we have quite as much as before

$$F(a, b) = \frac{\psi(a + b)}{\psi a \cdot \psi b} \ldots\ldots\ldots (7).$$

Change a into $(a + n)$ in (7);

$$\therefore F(a + n, b) = \psi(a + n + b) \div \{\psi(a + n) \cdot \psi b\},$$

whence we deduce $\dfrac{F(a, b)}{F(a + n, b)} = \dfrac{\psi(a + b) \cdot \psi(a + n)}{\psi a \cdot \psi(a + n + b)}$.

And since the right-hand member is not affected by exchanging n with b, Euler's well-known result follows:

$$\frac{F(a, b)}{F(a + n, b)} = \frac{F(a, n)}{F(a + b, n)}.$$

whence by equation (1) we have

$$F(a, b) = \frac{F(n, a) \,.\, F(n + a, b)}{F(n, a + b)}.$$

Make n infinite in comparison with a and b, and observe that by (4) we then find $F(n + a, b) = F(n, b)$: consequently we get

$$F(a, b) = \frac{F(n, a) \,.\, F(n, b)}{F(n, a + b)}, \text{ when } n = \infty \,.\,.\,.\,. \; (8).$$

This equation has a close analogy with (7), and leads us to examine the nature of the function $F(n, b)$ when $n = \infty$.

Put $x = n^{-1} y$; therefore

$$F(n + 1, a) = \int_0^1 (1 - x)^n \, x^{a-1} \, dx = \int_0^n (1 - n^{-1} y)^n \, n^{-a} \, y^{a-1} \, dy$$

$$= n^{-a} \,.\, \int_0^\infty e^{-y} \, y^{a-1} \, dy.$$

Denoting therefore the last integral by Γa, we obtain from equation (8) by mere substitution,

$$F(a, b) = \frac{\Gamma a \,.\, \Gamma b}{\Gamma (a + b)} \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,. \; (9).$$

Comparing (7) and (9), it occurs to inquire, what is the relation between ψ and Γ. The one *might* be the reciprocal of the other, as far as these two equations are concerned. Put then $\Gamma a = \chi a \,.\, (\psi a)^{-1}$, where χ is an unknown function. Substitute this value for Γ in (9) and simplify the result by (7). There remains $\chi (a + b) = \chi a \,.\, \chi b$; which can be solved either by differentiating or by still more elementary methods, and gives $\chi a = e^{\pm\gamma a}$, where γ is an unknown constant.

Hence $$\Gamma a = \frac{e^{-\gamma a}}{a} \,.\, \frac{e^{a}}{1 + a} \,.\, \frac{e^{\frac{1}{2}a}}{1 + \frac{1}{2} a} \,.\, \frac{e^{\frac{1}{3}a}}{1 + \frac{1}{3} a} \,.\, \&c. \,.\, (10),$$

and when a^2 is < 1,

$$\log \Gamma a = - \log a - \gamma a + \{a - \log (1 + a)\}$$
$$+ \{\tfrac{1}{2} a - \log (1 + \tfrac{1}{2} a)\} + \&c. \,.\,.\,.\,.$$
$$= - \log a - \gamma a + \tfrac{1}{2} S_2 a^2 - \tfrac{1}{3} S_3 a^3 + \tfrac{1}{4} S_4 a^4 - \&c. \,.\,.\,.\; (11).$$

Proceeding to the integral $\int_0^\infty e^{-x} \, x^{a-1} \, dx$, it is obvious from the first principles of integration that

$$\Gamma (a + 1) = a \, \Gamma a \,; \;\; \Gamma 1 = 1 \,; \;\; \Gamma n = 1 \,.\, 2 \,.\, 3 \,.\,.\, (n - 1) \,.\,.\,.\,. (12).$$

so that we readily find in (11) that

$$\left.\begin{aligned}\gamma &= (1 - \log 2) + (\tfrac{1}{2} - \log \tfrac{3}{2}) + (\tfrac{1}{3} - \log \tfrac{4}{3}) + \&c. \\ &= \tfrac{1}{2} S_2 - \tfrac{1}{3} S_3 + \tfrac{1}{4} S_4 - \tfrac{1}{5} S_5 + \&c.\end{aligned}\right\}\ldots(13).$$

Indeed from (11) it is easy to find $(1 - \gamma)$ by still more rapid convergence. But having sufficiently determined γ, let us take up the consideration of Γ from a new point of view. Namely, let us assume the equation (10) as the *definition** of Γ; then we know this to be identical with $\Gamma a = \int_0^\infty e^{-x} x^{a-1}\, dx$, as long as a is positive: so that all inferences drawn from the latter still hold good, under that limitation. Nevertheless, when a is negative, it is not the less true that $\Gamma(a + 1) = a\,\Gamma a$; which may be thus shewn.

Since $$\Gamma a = \frac{e^{-\gamma a}}{a} \cdot \frac{e^a}{1 + a} \cdot \frac{e^{\frac{1}{2}a}}{1 + \frac{1}{2}a} \cdot \frac{e^{\frac{1}{3}a}}{1 + \frac{1}{3}a} \cdot \&c.\ldots$$

change a into $-a$, and multiply by the result

$$\Gamma a \,.\, \Gamma(-a) = -a^{-2}\left\{(1 - a^2)(1 - \tfrac{1}{4}a^2)(1 - \tfrac{1}{9}a^2)\ldots\right\}^{-1}$$

$$= \frac{-\pi a^{-1}}{\sin . \pi a}.$$

Suppose a positive;

therefore $$\Gamma(-a) = \frac{-\pi}{\sin \pi a \,.\, \Gamma(1 + a)} \ldots\ldots (14).$$

Write $(a - 1)$ for a; then

$$\Gamma(1 - a) = \frac{\pi}{\sin \pi a \,.\, \Gamma a} = -a \,.\, \Gamma(-a):$$

and the last is an extension of $\Gamma(1 + a) = a\Gamma a$ to the case of a negative.

Then $\Gamma(1 + a)\,\Gamma(-a) = -\pi \div \sin \pi a$, for *all* values of a;

or $$\Gamma(1 - a) \,.\, \Gamma a = \pi \div \sin \pi a \ldots\ldots\ldots (15).$$

Thus the Complementary Equation is demonstrated for all values of a: and all other properties of Γ readily flow out of (11) (12) and (15).

* Then even if a is imaginary, it can be strictly proved that $\Gamma(1 + a) = a\,\Gamma a$, by taking, first, n factors of the series, and afterwards making $n = \infty$: also that $\Gamma(1 + a) \,.\, \Gamma(1 - a) = \pi a \div \sin \pi a$.