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Δυῶν ὀνομάτων μορφὴ μία.

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MACMILLAN, BARCLAY, AND MACMILLAN;
GEORGE BELL, LONDON;
HODGES AND SMITH, DUBLIN.
1848

illustrative of its equal power and utility: of these we select one in consequence of the historical interest connected with the circumstances attending it, and thus proceed to one of the expressed objects, though not perhaps the most useful part, of the present paper: in the first place, to establish a theorem connected with M. Jacobi's method of generating surfaces of the second order, which, so far as the author is aware, has not hitherto been proved, and which moreover, so far as he can see, though it appears immediately from the present principle, would be rather difficult to establish by any of the ordinary methods: and in the next place, to shew that the method of generation proposed by that illustrious mathematician, for the purpose of establishing a particular and interesting analogy between curves and surfaces of the second order, is implicitly contained in Professor MacCullagh's, or the modular method of generating the same surfaces.

[*To be continued.*]

Trinity College, Dublin, February 1847.

ON THE VALUES OF A PERIODIC SERIES AT CERTAIN LIMITS.

By Francis W. Newman.

The discussions which from time to time arise concerning Fourier's theorem and its simplest cases, shew that it is not yet superfluous to exhibit elementary and rigorous proof of these. The following process is that of Fourier himself, except that he has left it to his reader to apply it at the limits themselves. It seems instructively to shew how erroneous it is to assert, that in the algebra, "what is true *within* the limits is true *at* the limits."

Let two cardinal series be considered,

$$\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \tfrac{1}{7}\cos 7x + \&c.\ldots.$$

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \&c.\ldots.$$

which shall first be treated as containing a finite number of terms. Put Y, Z for the sums of $2n$ terms; and after it shall have been ascertained that the series converge when $n = \infty$, let y, z represent the sums *ad infinitum*.

$$\text{Then } \frac{dY}{dx} = -\{\sin x - \sin 3x + \sin 5x - \ldots. - \sin(4n-1)x\}$$

$$= \frac{\sin 4nx}{2\cos x}, \text{ by ordinary summation,}$$

and $\frac{dZ}{dx} = \cos x - \cos 2x + \cos 3x - \ldots - \cos 2nx,$

$$= \frac{\cos \frac{1}{2} x - \cos \frac{1}{2}(4n+1)x}{2 \cos \frac{1}{2} x}.$$

Integrating both by parts, we get

$$Y = C_n - \frac{\cos 4nx}{4n \,.\, 2 \cos x} + \frac{1}{8n} \int \cos 4nx \,.\, d \sec x,$$

$$Z = c_n + \tfrac{1}{2}x - \frac{\sin \frac{1}{2}(4n+1)x}{(4n+1)\cos \frac{1}{2}x} + \frac{1}{4n+1} \int \sin \frac{4n+1}{2} x \, d \sec \tfrac{1}{2}x.$$

Whenever the limits are such that the closing integrals are finite, the two terms vanish upon supposing $n = \infty$. But when $\pm x$ reaches $\frac{1}{2}\pi$ in the former or π in the latter, we cannot be sure of this: hence the investigation separates itself into cases.

First, in reference to Y, let ϵ be some positive finite number, however small, and x lie between $\pm(\frac{1}{2}\pi - \epsilon)$; then $\sec x$ is never greater than $\operatorname{cosec} \epsilon$, and $\int \cos 4nx \, d \sec x$ is finite between the finite limits. On the other hand, if we fix 0 as the lower limit of x in $\int$, since 0 is between $\pm(\frac{1}{2}\pi - \epsilon)$, we get

$$Y \text{ or } 1 - 3^{-1} + 5^{-1} - \ldots - (4n-1)^{-1} = C_n - (8n)^{-1} \cos 4nx;$$

which gives a finite value to C_n when $n = \infty$.

This finite value is

$$C = 1 - 3^{-1} + 5^{-1} - 7^{-1} + \&c. \ldots ad\ infin. = \tfrac{1}{4}\pi.$$

Make therefore $n = \infty$, and confine x between $\pm(\frac{1}{2}\pi - \epsilon)$, and we get

$$y = \tfrac{1}{4}\pi \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But this fails of giving us y when $x = \pm\frac{1}{2}\pi$, at which limit Fourier asserts that y has any or every value between $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$. A proof may be offered as follows.

Put $x = \frac{1}{2}\pi - \frac{u}{4n}$, and let u vary from a to 0.

Since $\frac{dY}{dx} = \frac{\sin 4nx}{2 \cos x}$, $\quad \therefore\ Y = C_n' + \int \frac{\sin 4nx}{2 \cos x} dx.$

Now $\cos x = \sin \frac{u}{4n}$; $dx = -\frac{du}{4n}$; $\sin 4nx = \sin(2n\pi - u) = -\sin u$;

therefore $$Y = C_n' + \tfrac{1}{2}\int \sin u \left(\sin \frac{u}{4n}\right)^{-1} . \frac{du}{4n}.$$

To fix C_n', take $u = 0$ as the lower limit of the integral, $x = \frac{1}{2}\pi$, $Y = (2n - 1)$ terms of a series of which every one vanishes; or $Y = 0$. Then $C_n' = 0$.

Next, make $n = \infty$;

therefore $$y = \tfrac{1}{2}\int_0 \frac{\sin u \,.\, du}{u} \,\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The range of u is up to $u = a$; but a is as arbitrary as u. At the extreme values $a = \pm\infty$, we get $y = \pm\frac{1}{4}\pi$; and by assigning intermediate values to u, we may give to y any proposed value between $\pm\frac{1}{4}\pi$. This justifies Fourier's assertion, that the locus of the curve

$$y = \cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \&c.\ldots \textit{ad infin.}$$

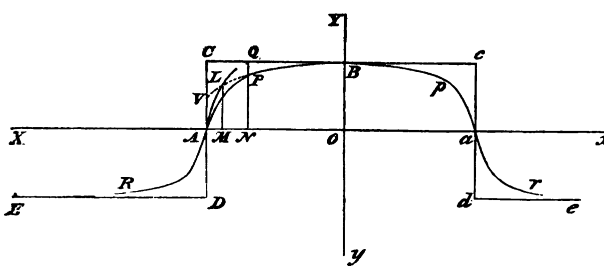
is a connected series of straight lines alternately parallel and perpendicular to the axes.

The explanation of this is given (from Fourier) in that well-known work, De Morgan's *Differential Calculus*.

Imagine the locus of

$$Y = \cos x - \tfrac{1}{3}\cos 3x + \ldots - (4n - 1)^{-1}\cos(4n - 1)\,x,$$

to be constructed, which cannot much differ from the line



RAPBpar, if *XOx*, *YOy* are the axes. Make $n = \infty$, and this locus approximates towards the system of lines *EDCcde* ....which is the true locus of the infinite equation that connects y and x.

This also shews *why* the whole perpendicular line *CD* is included in the locus. Let $ON = x$, $NP = Y$, $NQ = y$. If then we suppose x to be constant, and n to increase perpetually, P runs up towards Q; and if n is constant, but AN diminishes perpetually, P runs towards the single point A. But besides these hypotheses, we are at liberty *to suppose*

AN while vanishing, to be a function of n which is increasing; just as above, we have put $x = \frac{1}{2}\pi - \frac{1}{4}n^{-1}u$. Then P changes with n on to another curve, while N changes, with x, to another point M; or by the double change, P moves as to L, within the area $CAPQ$; and with the progressive increase of n and of x, it may tend towards *any* point V in the line AC. What point shall be V, depends on the function which AN is of n; or, with us, on the highest limit of u.

This may warn us generally, that in estimating the value of a series $y_n = \phi_1 x + \phi_2 x + \ldots + \phi_n x$, taken when $n = \infty$, the result, at singular points, depends on the question, whether x can itself be a function of n. Algebraically and in the abstract this must be looked on as possible: in each physical application the possibility will need to be inquired into.

Next, we may more rapidly dispatch the investigation of Z. Let the integral $\int \sin \frac{1}{2}(4n+1)\, x\, d \sec \frac{1}{2} x$, begin from $x = 0$; then $Z = 0$, since the number of terms is finite, and each separately $= 0$; consequently $c_n = 0$. Farther, let ε be positive, and x lie between $\pm(\pi - \varepsilon)$, which contains the value $x = 0$ by which we defined c_n. Then the unknown integral is finite. Put $n = \infty$, and we have

$$z = \tfrac{1}{2}x, \text{ as long as } x \text{ is between } \pm(\pi - \varepsilon) \ldots \quad (3).$$

But for the limit $x = \pm\,\pi$, we proceed otherwise.

$$\text{Put } x = \pi - \frac{v}{2n}, \quad Z = c_n' + \tfrac{1}{2}x - \tfrac{1}{2}\int \cos \frac{4n+1}{2}\, x \sec \tfrac{1}{2}x\, dx,$$

$$\cos \tfrac{1}{2}x = \sin \frac{v}{4n}, \quad \cos \frac{4n+1}{2}\, x = \sin\left(v + \frac{v}{4n}\right); \quad dx = -\frac{dv}{2n};$$

$$\text{whence} \quad Z = c_n' + \tfrac{1}{2}x + \int \sin\left(v + \frac{v}{4n}\right)\left(\sin \frac{v}{4n}\right)^{-1} \cdot \frac{dv}{4n}.$$

Let the last integral begin from $v = 0$, $x = \pi$, therefore as n is finite, $Z = 0$, and $0 = c_n' + \frac{1}{2}\pi$; afterwards pass to the limit of $n = \infty$; therefore

$$z = \frac{x - \pi}{2} + \int_0 \frac{\sin v dv}{v};$$

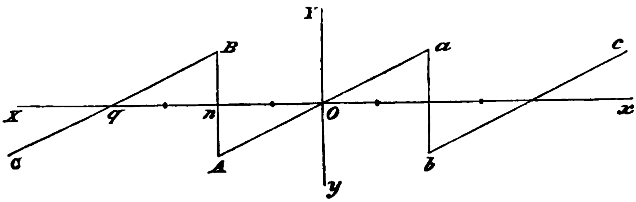
or, since x converges to π, when n increases towards infinity,

$$z = \int_0 \frac{\sin v dv}{v}, \text{ when } x \text{ is at } \pi \ldots\ldots\ldots (4).$$

This, as before, means that z has any value whatever between $\pm \frac{1}{2}\pi$, and shews the locus of

$$y = \sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \&c.$$

to be of the zigzag form. . . . *CABabc*. . . .



ON A GENERAL TRANSFORMATION OF ANY QUANTITATIVE FUNCTION.

By George Boole.

There is a view of the general problem of the solution of equations algebraic and transcendental, which I do not remember to have seen taken by any writer on the subject. It is embodied in the following proposition.

If we knew all the transformations of the arbitrary function $f(x)$, we should be able to solve any proposed equation by a general theorem.

This will be obvious if we consider that the solution of an equation is, properly speaking, the transformation of an inverse function into a function in which all the implied operations are direct. Thus the solution of the equation $\phi(u) = x$ is really a transformation of the inverse function $\phi^{-1}(x)$. Lagrange's and Laplace's theorems might be interpreted into results of this nature, although they are generally regarded from a different point of view. I propose here to consider the more general question which is suggested by the above considerations. The most remarkable conclusion to which my inquiries lead is, that the general transformation of $f(x)$ involves a new arbitrary function, particular forms of which conduct us to the theorems of Laplace and Lagrange. The analysis which I shall employ is, in principle, that which I have adopted in a similar investigation, published in the 21st volume of the *Transactions of the Royal Irish Academy*, but it is more simple and more direct.