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ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.

By Francis W. Newman.

§. I.

1. The general formula $\int F_1x\,.\log F_2x\,.dx$, where F_1F_2 denote rational functions, contains a variety of integrals, all of which, it will be shewn, can be reduced to *three*.

By the common method of finding $\int F_1x\,.dx$, we perceive that there is some rational function F_3 which fulfils the equation

$$F_1x = \frac{d}{dx}F_3x + \Sigma\frac{A}{x-e} + \Sigma\frac{px+q}{(x-\mu)^2+\nu^2}.$$

Also, if F_2x be reduced to the form of a single algebraic fraction, it may be denoted by $F'x \div F''x$, where F' and F'' are each *integer*. Consequently we may write

$$\log F_2x = \Sigma.A_1\log(ax+b) + \Sigma.A_2\log(a'x^2+b'x+c').$$

It immediately follows that $\int F_1x\,.\log F_2xdx$ is separable into the two forms $\int F_1x.\log(ax+b)\,dx$ and $\int F_1x.\log(a'x^2+b'x+c')\,dx$. In the former, introduce the preceding value of F_1x, and we obtain for the integral

$$\log(ax+b)\,.\,F_3x - \int\frac{F_3x.adx}{ax+b}$$
$$+\ \Sigma.A\int\frac{\log(ax+b)}{x-e}dx + \Sigma\int\log(ax+b)\,.\,\frac{(px+q)\,dx}{(x-\mu)^2+\nu^2}.$$

Of the three integrals which here appear, the first is rational. In the second assume $ax+b=mx'$; $\therefore a(x-e)=mx'-b-ae$. Assume farther, $m=b+ae$; then

$$\int\frac{\log(ax+b)\,dx}{x-e} = \log m\,.\int\frac{dx}{x-e} + \int\frac{\log x'.\,dx'}{x'-1},$$

provided that m, or $(b+ae)$, is positive. If otherwise, put $x=e+m'x''$, and $am'=-(b+ae)$;

$$\therefore\ \int\frac{\log(ax+b)\,dx}{x-e} = \log(ax+b)\,.\log\frac{x-e}{m'} - \int\frac{\log x''.\,dx''}{x''-1}.$$

In either case we arrive at the elementary form

$$L(x) = \int_1\frac{\log x.dx}{x-1}\ \ldots\ldots\ldots\ldots\ (1),$$

which Spence has tabulated. As for the integral

$$\int\log(ax+b)\,.\,\frac{(px+q)\,dx}{(x-\mu)^2+\nu^2},$$

the same assumption, $ax + b = mx'$, if we give to m a suitable constant value, produces two general forms which may be denoted by

$$\int \frac{\log x dx}{X} \text{ and } \tfrac{1}{2}\int \log x \, d \log X; \quad \text{if } X = x^2 - 2x\cos a + 1.$$

Let ω be an arc such that $\tan \omega = x \sin a \div (1 - x \cos a)$, or, what is the same, $x = \sin \omega \div \sin(\omega + a)$: then

$$d\omega = \frac{\sin a . dx}{X}; \text{ and } \sin a . \int \frac{\log x dx}{X}$$

$$= \int \log \sin \omega d\omega - \int \log \sin(\omega + a) d\omega.$$

Suppose ♄ to be a symbol for a new function, such that

$$♄(\omega) = -\int_0 \log \sin \omega . d\omega \quad \ldots\ldots\ldots\ldots (2);$$

then $$\sin a . \int_0 \frac{\log x dx}{X} = ♄(\omega + a) - ♄\omega - ♄a \ldots\ldots (2).$$

No similar reduction occurs, by which we can exterminate the arbitrary constant from the next integral; and we must be satisfied with writing

$$\Lambda(x, a) \text{ for } \int_0 \frac{\log x . (x - \cos a) \, dx}{x^2 - 2x \cos a + 1} \text{ or } \tfrac{1}{2}\int lx . dlX \ldots (3).$$

It will be sometimes convenient to put

$$\lambda(x, a) \text{ for } \tfrac{1}{2}\int_0 \log(x^2 - 2x \cos a + 1) . \frac{dx}{x} \ldots\ldots (4),$$

which is a supplemental function to Λ, and so related that

$$\Lambda(x, a) + \lambda(x, a) = \tfrac{1}{2} \log x . \log X.$$

We may write Λx, λx when no change of a is contemplated.

2. We have now to go back to $\int F_1 x . \log(ax^2 + bx + c) \, dx$. By substituting as before for F_1, we reduce the integral to

$$F_2 x . \log(ax^2 + bx + c) - \int F_2 x . \frac{(2ax + b) \, dx}{ax^2 + bx + c}$$

$$+ \Sigma A \int \frac{\log(ax^2 + bx + c)}{x - e} dx + \Sigma \int \log(ax^2 + bx + c) . \frac{(px + q) \, dx}{(x - \mu)^2 + \nu^2}.$$

Of the three integrals remaining, the first is rational. The second is readily reduced to the form λ, by making $(x - e) = mx'$. The third, by making $x - \mu = mx'$, and determining m aright, produces the two new forms

$$X_1 = \int \log X . \frac{n dx}{x^2 + n^2}; \quad X_2 = \int \log X . \frac{x dx}{x^2 + n^2};$$

each of which has two arbitrary constants, a and n. But fortunately we can reduce X_1 to ϟ, and X_2 to L or λ. First, for X_1, put $x = n \tan \omega$, $n = \tan \nu$; $\dfrac{ndx}{x^2 + n^2} = d\omega$.

$$X = 1 - 2n \tan \omega \cos a + n^2 \tan^2 \omega$$
$$= (\cos^2 \omega - 2n \sin \omega \cos \omega \cos a + n^2 \sin^2 \omega) \div \cos^2 \omega$$
$$= \left\{(1 + n^2) - 2n \sin 2\omega \cos a + (1 - n^2).\cos 2\omega\right\} \div 2 \cos^2 \omega$$
$$= (1 - \sin 2\nu \sin 2\omega \cos a + \cos 2\nu.\cos 2\omega) \div 2 \cos^2 \nu.\cos^2 \omega.$$

Let μ, β be taken such that $\left.\begin{matrix}\sin \mu \sin \beta = \sin 2\nu \cos a \\ \sin \mu \cos \beta = \cos 2\nu\end{matrix}\right\}$;

$\therefore\ \cos \mu = \sin 2\nu \sin a$, and $\tan \beta = \tan 2\nu.\cos a$.

Also $X = \{1 + \sin \mu (\cos 2\omega \cos \beta - \sin 2\omega \sin \beta)\} \div 2 \cos^2 \nu.\cos^2 \omega$,
$= (1 + \sin \mu \cos \theta) \div 2 \cos^2 \nu \cos^2 \omega$; if $\theta = 2\omega + \beta$:

whence $X_1 = \int \log X.d\omega = \frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$
$- \omega \log (2 \cos^2 \nu) - 2\text{ϟ}(\frac{1}{2}\pi - \omega)$....(5):

in which the remaining integral has but one arbitrary constant.

Farther, let $m = \tan \frac{1}{2}\mu$, or $\sin \mu = 2m \div (1 + m^2) = 2m \cos^2 \frac{1}{2}\mu$;
$\therefore \log (1 + \sin \mu \cos \theta) = \log (1 + 2m \cos \theta + m^2) + 2 \log \cos \frac{1}{2}\mu$.

Assume η such that $\tan \eta = \sin \theta \div (m + \cos \theta)$,
or $m = \sin (\theta - \eta) \div \sin \eta$,

$$\therefore 1 + 2m \cos \theta + m^2 = \sin^2 \theta + (m + \cos \theta)^2$$
$$= \sin^2 \theta + \left(\frac{\sin \theta}{\tan \eta}\right)^2 = \left(\frac{\sin \theta}{\sin \eta}\right)^2.$$

whence $\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$
$$= \int \{\log \sin \theta - \log \sin \eta + \log \cos \tfrac{1}{2}\mu\}\, d\theta$$
$$= -\text{ϟ}\theta - \int \log \sin \eta.d\theta + \theta.\log.\cos \tfrac{1}{2}\mu.$$

Now $\int \log \sin \eta.d\theta$
$$= \int \log.\frac{\sin (\theta - \eta)}{m}.d\theta = \int \log \frac{\sin (\theta - \eta)}{m}.\{d(\theta - \eta) + d\eta\}$$
$$= -\text{ϟ}(\theta - \eta) - (\theta - \eta) \log m + \int \log \sin \eta.d\eta$$
$$= -\text{ϟ}(\theta - \eta) - (\theta - \eta) \log \tan \tfrac{1}{2}\mu - \text{ϟ}\eta.$$

Therefore
$$\left.\begin{matrix}\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta \\ = \text{ϟ}(\theta - \eta) + \text{ϟ}\eta - \text{ϟ}\theta + \theta \log \sin \frac{1}{2}\mu - \eta \log \tan \frac{1}{2}\mu\end{matrix}\right\}\ldots(6),$$

which is a general formula, provided that $\tan \eta = \dfrac{\sin \theta}{\tan \frac{1}{2}\mu + \cos \theta}$;

and completes the reduction of X_1 to the function ϟ.

3. The integral X_2 remains. Using for X the same transformation as before, let us write 2θ in place of θ, so that now $2\theta = 2\omega + \beta$. We have, moreover,

$$\frac{xdx}{x^2+n^2} = \tfrac{1}{2}\, d\log(n^2+x^2) = \tfrac{1}{2}\, d\log\sec^2\omega = \tan\omega d\omega$$

$$\text{and also} = -\tfrac{1}{2}\, d\log(2\cos^2\nu\,.\cos^2\omega).$$

Whence

$$X_2 = \int\{\log(1+\sin\mu\cos 2\theta) - \log(2\cos^2\nu\,.\cos^2\omega)\}\frac{xdx}{x^2+n^2}$$

$$= \int\log(1+\sin\mu\cos 2\theta)\tan\omega d\omega + \tfrac{1}{4}\log^2.(2\cos^2\nu\,.\cos^2\nu).$$

Put $b = \tan\frac{1}{2}\beta$; $t = \tan\theta$, $d\omega = d\theta = \dfrac{dt}{1+t^2}$:

$$\cos 2\theta = \frac{1-t^2}{1+t^2};\ \tan\omega = \tan\left(\theta - \frac{b}{2}\right) = \frac{t-b}{1+bt};$$

$$\text{and } \tan\omega.d\omega = \frac{t-b}{1+bt}\cdot\frac{dt}{1+t^2} = \frac{tdt}{1+t^2} - \frac{bdt}{1+bt}.$$

Hence the integral which remains, becomes

$$\left\{\int\log\{1+t^2+\sin\mu\,.(1-t^2)\} - \log(1+t^2)\right\}.\left(\frac{tdt}{1+t^2} - \frac{bdt}{1+bt}\right).$$

Write

$T_1 = \int\log\{1+\sin\mu+(1-\sin\mu)\,.\,t^2\}\,.\,\frac{1}{2}d\log(1+t^2)$,

$T_2 = \int\log(1+t^2)\,.\,\frac{1}{2}d\log(1+t^2) = \frac{1}{4}\log^2(1+t^2) = \log^2\cos\theta$,

$T_3 = \int\log\{1+\sin\mu+(1-\sin\mu)\,.\,t^2\}\,d\log(1+bt)$,

$T_4 = \int\log(1+t^2)\,d\log(1+bt)$.

Then $X_2 = T_1 - T_2 - T_3 + T_4 + \frac{1}{4}\log^2.(2\cos^2\nu\,.\cos^2\omega)$.

To find T_1, let $1+t^2 = mv$, and $m = 2\sin\mu \div (1-\sin\mu)$;

$\therefore\quad T_1 = \frac{1}{2}\int l\{2\sin\mu\,.(1+v)\,dl(mv)$

$$= \tfrac{1}{2}l(2\sin\mu)\,l(mv) + \tfrac{1}{2}L(1+v);$$

where $mv = \sec^2\theta$,

$$1+v = \frac{1+t^2+\sin\mu(1-t^2)}{2\sin\mu} = \frac{\sec^2\theta(1+\sin\mu\cos 2\theta)}{2\sin\mu};$$

so that $T_1 = -\log(2\sin\mu)\log\cos\theta + \frac{1}{2}L\dfrac{1+\sin\mu\cos 2\theta}{\sin\mu(1+\cos 2\theta)}$.

For T_3, put $1+bt = kz$, $1+\sin\mu+(1-\sin\mu)\,t^2$

$= b^{-2}.\{1+b^2-(1-b^2)\sin\mu - 2kz(1-\sin\mu) + k^2z^2(1-\sin\mu)\}$.

Take k such that $k^2(1-\sin\mu) = 1+b^2-(1-b^2)\sin\mu$;

or $= (1+b^2)\{1-\cos\beta\sin\mu\} = \sec^2\frac{1}{2}\beta(1-\cos 2\nu)$;

$$\therefore \quad k = \sec \frac{\beta}{2} \sqrt{\frac{1 - \cos 2\nu}{1 - \sin \mu}}.$$

Also let $\cos \gamma = k^{-1} = \cos \dfrac{\beta}{2} \sqrt{\dfrac{1 - \sin \mu}{1 - \cos 2\nu}}$,

and observe that $b^{-2}k^2 (1 - \sin \mu) = (\sin \frac{1}{2}\beta)^{-2}. (1 - \cos 2\nu)$,

also $kz = 1 + bt = 1 + \tan \frac{1}{2}\beta \tan \theta = \dfrac{\cos (\theta - \frac{1}{2}\beta)}{\cos \frac{1}{2}\beta \cos \theta} \propto \dfrac{\cos \omega}{\cos \theta}$.

Hence

$$T_3 = \int \log \{(\sin \tfrac{1}{2}\beta)^{-2}. (1 - \cos 2\nu).(1 - 2z \cos \gamma + z^2)\}\, d \log (kz)$$

$$= \log \{(\sin \tfrac{1}{2}\beta)^{-2}. (1 - \cos 2\nu)\} \log \frac{\cos \omega}{\cos \theta} + 2\lambda (z, \gamma).$$

From this we may deduce T_4 by momentarily supposing $\mu = 0$, which makes $\cos 2\nu = 0$; so that, writing $k'y$ for kz, we get $k' = \sec \frac{1}{2}\beta$, and γ changes into $\frac{1}{2}\beta$. Also $y = \dfrac{\cos \omega}{\cos \theta}$.

$$\therefore \quad T_4 = - \log \sin^2 \tfrac{1}{2}\beta \log \frac{\cos \omega}{\cos \theta} + 2\lambda (y, \tfrac{1}{2}\beta),$$

and $- T_3 + T_4$

$$= - \log (1 - \cos 2\nu) \log \frac{\cos \omega}{\cos \theta} + 2\lambda (y, \tfrac{1}{2}\beta) - 2\lambda (z, \gamma);$$

in which we may deduce z, γ from $y, \frac{1}{2}\beta$ by writing

$$c^2 = \frac{1 - \sin \mu}{1 - \cos 2\nu}, \quad z = cy, \quad \cos \gamma = c. \cos \tfrac{1}{2}\beta.$$

Combining all the results, we have to observe that (neglecting constants)

$$\tfrac{1}{4}l^2 (2 \cos^2 \nu \cos^2 \omega) - l (2 \sin \mu)\, l \cos \theta - l^2 \cos \theta$$
$$- l (1 - \cos 2\nu) (l \cos \omega - l \cos \theta)$$

$$= \log^2 (n \sec \omega) - \log^2 \left(\cos \theta . \frac{\sqrt{\sin \mu}}{\sin \nu}\right).$$

Whence, finally,

$$\left.\begin{aligned} X_2 = \log^2 (n \sec \omega) - \log^2 \left(\cos \theta . \frac{\sqrt{\sin \mu}}{\sin \nu}\right) \\ + \tfrac{1}{2} L \frac{1 + \sin \mu \cos 2\theta}{\sin \mu\, (1 + \cos 2\theta)} + 2\lambda \left(y, \frac{\beta}{2}\right) - 2\lambda (z, \gamma) \end{aligned}\right\} \ldots (7).$$

Observe that $ny^{-1} = n \cos \frac{1}{2}\beta - x \sin \frac{1}{2}\beta$; and the quantity under L may also be denoted by $\dfrac{y^2 X \cos^2 \nu}{\sin \mu}$. The result thus obtained admits likewise of other forms, by means of the properties of λ and Λ; but all that is here aimed at, is to shew the possibility of the reduction.

It is easy to verify our result, in the case of $a = \frac{1}{2}\pi$. On the whole it has appeared that the integral $\int F_1 x \log F_2 x dx$ contains only three elementary forms, which we have denoted by L, ϟ, Λ. It is proposed to call these Logarithmic Integrals *of the Second Order*.

4. Before leaving the integrals X_1, X_2, it may be well to examine the special cases of $n = 1$, and of $x = \infty$. First, to find X_1 when $x = \infty$.

Put $X' = \int_0 \tan^{-1}\frac{x}{n} \cdot d \log X = \log X . \tan^{-1}\frac{x}{n} - X_1$:

$$\therefore \frac{dX'}{dn} = -\log X . \frac{x}{n^2 + x^2} - \frac{dX_1}{dn};$$

$$\text{and when } x = \infty, \ \frac{dX'}{dn} = -\frac{dX_1}{dn}.$$

Again, $\frac{dX'}{dn} = \int_0 \frac{-x}{n^2 + x^2} . d \log X$; which we assume

$$= \int_0 \left\{ \frac{2px + 2q}{x^2 + n^2} + \frac{2r(x - \cos a) + 2s \sin a}{x^2 - 2x \cos a + 1} \right\} dx;$$

and by common methods we find that if $N = n^4 + 2n^2 \cos 2a + 1$,

$$p = -r = \frac{\cos a . (n^2 + 1)}{N}; \ q = -\frac{n(n^2 + \cos 2a)}{N}; \ s = \frac{(n^2 - 1) \sin a}{N}.$$

$$\text{Also } \frac{dX'}{dn} = -p \log (n^2) + p \log \frac{x^2 + n^2}{X}$$

$$+ 2q \tan^{-1}\frac{x}{n} + 2s . \tan^{-1} . \frac{x \sin a}{1 - x \cos a}.$$

Let $x = \infty$; then integrating for n, observing that $\frac{dX'}{dn} = -\frac{dX_1}{dn}$,

$$-X_1 = -\int_0 2 \log n . pdn + \pi \int_0 qdn + 2(\pi - a) \int_0 sdn;$$

observing that as X_1 vanishes with n, no function of x is to be added. Now

$$2pdn = \frac{2 \cos a (n^2 + 1) dn}{n^4 + 2n^2 \cos 2a + 1} = \frac{\cos a . dn}{n^2 - 2n \sin a + 1} + \frac{\cos a \, dn}{n^2 + 2n \sin a + 1};$$

$$\therefore \text{ if } \tan \rho = \frac{n \cos a}{1 - n \sin a}, \text{ and } \tan \sigma = \frac{n \cos a}{1 + n \sin a}$$

$$\int_0 2 \log n . pdn = \int_0 \frac{\cos a . \log n . dn}{n^2 - 2n \sin a + 1} + \int_0 \frac{\cos a . \log n . dn}{n^2 + 2n \sin a + 1}$$

= [ϟ{ρ+(½π−a)} − ϟρ − ϟ(½π−a)] + [ϟ{σ+(½π+a)} − ϟσ − ϟ(½π+a)].

It will in a following section appear that

$$\text{ϟ}\left(\tfrac{1}{2}\pi - a\right) + \text{ϟ}\left(\tfrac{1}{2}\pi + a\right) = \text{ϟ}\pi.$$

Again, $\int_0 q\,dn = -\frac{1}{4}\log N$; $\int_0 2s\,dn = \frac{1}{2}\log . \dfrac{n^2 - 2n\sin a + 1}{n^2 + 2n\sin a + 1}$.

As before, take $\tan\beta = \tan 2\nu \cos a$, and $\cos\mu = \sin 2\nu \sin a$; add to this, $\tan\beta' = \cos 2\nu \tan a$; $\therefore \rho + \sigma = \beta$, $\rho - \sigma = a - \beta'$; from which we easily find ρ and σ. Also

$$\int_0 2s\,dn = \tfrac{1}{2}\log . \frac{1 - \sin 2\nu \sin a}{1 + \sin 2\nu \sin a} = \log \tan \tfrac{1}{2}\mu,$$

and $N = (n^2 + 1)^2 . (1 - \sin^2 2\nu . \sin^2 a) = \sec^4 \nu . \sin^2 \mu$:

so that finally,

$$\left.\begin{aligned} &\int_0^\infty \log\left(1 - 2n\tan\omega\cos a + n^2\tan^2\omega\right) d\omega \\ &= \tfrac{1}{2}\pi \log\left(\sec^2\nu . \sin\mu\right) - (\pi - a)\log\tan\tfrac{1}{2}\mu \\ &\quad + \text{ϟ}\left(\tfrac{1}{2}\pi + \rho - a\right) - \text{ϟ}\left(\tfrac{1}{2}\pi - \sigma - a\right) - \text{ϟ}\rho - \text{ϟ}\sigma \end{aligned}\right\} \ldots (8).$$

A similar process applies to X_2 when $x = \infty$; and by help of the property (to be hereafter proved) that, when $x = \infty$,

$$\left\{2\Lambda(x, a) - \log^2 x\right\} = \tfrac{2}{3}\pi^2 - 2\pi a + a^2;$$

yields

$$\left.\begin{aligned} &\left\{\log^2 x - X_2\right\} \text{ when } x = \infty, \\ &= \tfrac{2}{3}2\pi^2 - \pi a - \tfrac{1}{2}\pi\beta - (\pi - a)\beta' + \tfrac{1}{2}\Lambda\left(n^2, \pi - 2a\right) \end{aligned}\right\} \ldots (9).$$

Lastly:

When $n = 1$, $\dfrac{dX_2}{da} = \displaystyle\int_0 \frac{2x\sin a}{X} . \frac{x\,dx}{1 + x^2} = \tan a . \int_0 \left\{\frac{x}{X} - \frac{x}{1 + x^2}\right\} dx$

$$= \tan^{-1} . \frac{x\sin a}{1 - x\cos a} + \frac{\tan a}{2}\log\frac{X}{1 + x^2}.$$

Let $\cos a = h$, and observe that

$$\frac{d\lambda(x, a)}{da} = \int_0 \frac{\sin a . dx}{X} = \tan^{-1} . \frac{x\sin a}{1 - x\cos a};$$

$$\therefore X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}\int_0 \log\left\{1 - \frac{2xh}{1 + x^2}\right\}\frac{dh}{h}.$$

To find the arbitrary f, let $a = \frac{1}{2}\pi$, $h = 0$, $\therefore X_2 = \frac{1}{4}\log^2(1 + x^2)$,

$$\text{and } \lambda(x, a) = \tfrac{1}{2}\int_0 \log(1 + x^2)\frac{dx}{x} = \tfrac{1}{4}L(1 + x^2);$$

$$\left.\begin{aligned} &\therefore f(x) = \tfrac{1}{4}\log^2(1 + x^2) - \tfrac{1}{4}L(1 + x^2) \\ &\text{and } X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}L\left(\frac{X}{1 + x^2}\right) \end{aligned}\right\} \ldots . (10),$$

which is a simpler expression than would arise from putting $n = 1$ in equation (7).

§ II.—*On Spence's Integral* $\int_1 \frac{\log x\, dx}{x-1}$.

5. Spence has tabulated this integral, on the assumption that x is positive; and this suffices in practice. Yet it embarrasses us in generalizing concerning the integrals which are partially reducible to L, not to be at liberty to suppose x negative. Supposing log x to have arisen out of integration, and to be $= \int \frac{dx}{x}$, no imaginary quantity results from regarding x as negative: in fact, we may look on *log* x as a short mode of writing $\frac{1}{2} \log x^2$; then, in passing through 0, x produces no discontinuity in L.

The following are the chief properties of L, which are easily verified:

$$Lx + L(-x) = \tfrac{1}{2} L(x^2) + \tfrac{3}{2} L0,$$
$$L(\pm x) + L(1 \mp x) = \log x \,.\, \log(1 \mp x) + L0,$$
$$Lx + Lx^{-1} = \tfrac{1}{2} \log^2 x \quad (x \text{ positive}),$$
$$L(1+x) + L(1-x) = \tfrac{1}{2} L(1-x^2),$$
$$L(1+x) + L(1+x^{-1}) = \tfrac{1}{2} \log^2 x + C;$$

where $C = 2L2$, if x is positive; but $C = 2L0$, if x is negative. This is proved by making $x = 1$ in the former case, and $x = -1$ in the latter. The discontinuity is occasioned by $L(1 + x^{-1})$ becoming infinite, when x is passing through 0. So, if we wish to make x negative in the third formula, we must add $2L(-1)$ or $-\frac{1}{2}\pi^2$ on the right-hand side. Farther, we have

$$-L0 = 2L2 = \tfrac{1}{6}\pi^2, \quad L(-1) = -3L2 = -\tfrac{1}{4}\pi^2.$$

When $(x - 1)$ is infinitesimal,

$$Lx = x - 1, \quad \text{and } \tfrac{1}{4}\pi^2 + L(-x) = \left(\frac{x-1}{2}\right)^2.$$

When x is large,

$$L(-x+1) = 2L0 + \tfrac{1}{2}\log^2 x + 1^{-2}x^{-1} + 2^{-2}x^{-2} + 3^{-2}x^{-3} + 4^{-2}x^{-4} + \&c. \ldots$$

If we desire to know $L(-x)$ numerically, we may either calculate it by the last formula, or (when x is not large) deduce it by the first or second of the equations from Spence's Table.

In future I shall always employ log x as a mere representation of $\int \frac{dx}{x}$ or $\frac{1}{2}\log(x^2)$; and it will only be necessary, in correcting integrals, to observe whether the arbitrary constant is altered by supposing the quantity under *log* to pass from positive to negative.

§. III.—*On the integral*, $-\int_0 \log \sin x dx$.

6. Since log sin x and log sin $(-x)$ are by hypothesis the same, or to speak otherwise, since $ϟ(x) = -\frac{1}{2}\int_0 \log \sin^2 x\, dx$,

$$\therefore \quad ϟ(-x) = -ϟx \ldots\ldots\ldots\ldots\ldots (11).$$

Also $ϟ(n\pi \pm x) = \mp \int \log \sin (n\pi \pm x)\, dx = \mp \int \log \sin x\, dx$,

$$\text{or} \quad ϟ(n\pi \pm x) = ϟ(n\pi) \pm ϟx.$$

Make n successively 1, 2, 3,... and $x = \pi$ } and we find $ϟ(n\pi) = nϟ\pi$.

Hence it readily follows that

$$\left.\begin{array}{c} ϟ(n\pi \pm x) = nϟ\pi \pm ϟx \\ ϟ(\pi - x) = ϟ\pi - ϟx; \quad 2ϟ\tfrac{1}{2}\pi = ϟ\pi \end{array}\right\} \ldots\ldots (12).$$

These equations indicate, that to tabulate ϟ from $x = 0$ to $x = \frac{1}{2}\pi$ will suffice.

7. To find $ϟ\pi$.

Since $-\log (2 \sin x)$

$$= \cos 2x + 2^{-1} \cos 4x + 3^{-1} \cos 6x + 4^{-1} \cos 8x + \&c.,$$

therefore $ϟx = x \log 2$

$$+ \tfrac{1}{2}\{1^{-2} \sin 2x + 2^{-2} \sin 4x + 3^{-2} \sin 6x + \&c.\ldots\} \ldots (13).$$

Hence $\quad ϟ\pi = \pi \log 2 = 2{\cdot}177586\ 0933046.$

Also $\quad ϟ\tfrac{1}{4}\pi = \tfrac{1}{4}ϟ\pi + \tfrac{1}{2}\{1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + 9^{-2} - \&c.\}$.

8. Since $\sin 2x = 2 \sin x \sin (\frac{1}{2}\pi - x)$, take logs. and integrate;

$$\therefore \quad \tfrac{1}{2}ϟ(2x) = (\tfrac{1}{2}\pi - x) \log 2 + ϟx - ϟ(\tfrac{1}{2}\pi - x) \ldots (14).$$

We may generalize this theorem. Since

$$\sin nx = 2^{n-1}.\sin x.\sin\left(\frac{\pi}{n} + x\right).\sin\left(\frac{2\pi}{n} + x\right)\ldots\sin\left(\frac{n-1}{n}\pi + x\right),$$

take the logarithms, as before, and integrate;

$$\therefore \quad \frac{1}{n}ϟ(nx) = C - (n-1)\, x \log 2 + ϟx + ϟ\left(\frac{\pi}{n} + x\right) + \ldots$$
$$+ ϟ\left(\frac{n-1}{n}\pi + x\right):$$

To find C, make $x = 0$;

$$\therefore \quad -C = ϟ\frac{\pi}{n} + ϟ\frac{2\pi}{n} + \ldots + ϟ\frac{n-1}{n}\pi.$$

In inverted order,

$$-C = \daleth\frac{n-1}{n}\pi + \daleth\frac{n-2}{n}\pi + \ldots + \daleth\frac{\pi}{n}.$$

Add these together, observing that

$$\daleth\left(\frac{r\pi}{n}\right) + \daleth\left(\frac{n-r}{n}\pi\right) = \daleth\pi;$$

$$\therefore\ -2C = (n-1)\daleth\pi = (n-1)\pi\log'2,$$

whence $\frac{1}{n}\daleth(nx) = -(n-1)(\frac{1}{2}\pi + x)\,l2$

$$+\daleth x + \daleth\left(\frac{\pi}{n} + x\right) + \ldots + \daleth\left(\frac{n-1}{n}\pi + x\right)\ldots.(15).$$

If we change x to $-x$, remembering (11),

$$\frac{1}{n}\daleth(nx) = +(n-1)(\tfrac{1}{2}\pi - x)\,l2 + \daleth x - \daleth\left(\frac{\pi}{n} - x\right) - \&c.\ldots$$

$$-\daleth\left(\frac{n-1}{n}\pi - x\right),$$

which contains (14) as a particular case.

From either of them, by help of (12), putting $n = 3$ and $n = 5$,

$$\left.\begin{aligned}\tfrac{1}{3}\daleth(3x) &= -2xl2 + \daleth x + \daleth(60^\circ + x) - \daleth(60^\circ - x)\\ \tfrac{1}{5}\daleth(5x) &= -4xl2 + \daleth x + \daleth(36^\circ + x) - \daleth(36^\circ - x)\\ &\qquad + \daleth(72^\circ + x) - \daleth(72^\circ - x)\end{aligned}\right\}\ldots(16).$$

If in (14) we make $x = 30^\circ$, and in the former equation of (16) make $x = 15^\circ$, we get, by help of (14),

$$\left.\begin{aligned}\tfrac{3}{2}\daleth\,60^\circ &= \daleth\,30^\circ + \tfrac{1}{3}\daleth\pi\\ \tfrac{4}{3}\daleth\,45^\circ &= 2\daleth\,15^\circ - \tfrac{1}{2}\daleth\,30^\circ + \tfrac{1}{4}\daleth\pi\end{aligned}\right\}\ldots\ldots(17).$$

9. By help of equation (14), if a table of $\daleth$ has been computed from $x = 0$ to $x = 45^\circ$, we can continue it to $x = 90^\circ$.

Generally, if the table be given from $x = 0$ to $x = a$, we can work by a double process, as follows. First, suppose $2x$ to vary from 0 to a, in which case $\daleth x$ and $\daleth(2x)$ being known, we determine $\daleth(90^\circ - x)$ by equation (14): thus $\daleth x$ becomes known from $x = 90^\circ$ to $x = 90^\circ - \frac{1}{2}a$.

Next, let $2x$ vary within the last-named limits, *supposing a to be not less than* 45°, and x may lie within the limits $x = 0$, $x = a$; thus $\daleth(2x)\ \daleth(x)$ are again known; and we deduce $\daleth(\frac{1}{2}\pi - x)$; that is, we find $\daleth x$ from $x = \frac{1}{4}\pi$ to $x = \frac{1}{4}(\pi + a)$. Let $a_2 = \frac{1}{4}(\pi + a)$; and the process may be repeated, writing a_2 for a; then we fill the table as *high* as $x = \frac{1}{4}(\pi + a_2) = a_3$, and as *low* as $x = \frac{1}{2}(\pi - a_2)$. Again, let $a_4 = \frac{1}{4}(\pi + a_3)$; and, by a third process, we rise as high as $x = a_4$ and come down as low as $x = \frac{1}{2}(\pi - a_3)$; and so on.

Now $a_3 = 4^{-1}\pi + 4^{-2}(\pi + a)$; $a_4 = 4^{-1}\pi + 4^{-2}\pi + 4^{-3}(\pi + a)$; &c.

Ultimately $a_\infty = \pi\,\{4^{-1} + 4^{-2} + 4^{-3} + 4^{-4} + \ldots\} = \frac{1}{3}\pi$,

and $$\tfrac{1}{2}(\pi - a_\infty) = \tfrac{1}{2}(\pi - \tfrac{1}{3}\pi) = \tfrac{1}{3}\pi.$$

Thus the opposite series meet, and the table is filled.

In practice, if x in the table passes from degree to degree, the steps will be as follows. Given the table up to $x = 45°$.

First; let $x = 1°, 2°..22°$; and, by (14), fill the table from $x = 89°$ to $x = 68°$.

Next; let $x = 34°, 35°..44°$; and fill from 56° to 46°.

Thirdly; let $x = 23°, 24°..28°$; and fill from 67° to 62°.

Fourthly; let $x = 31°, 32°, 33°$; and fill from 59° to 57°.

Fifthly; let $x = 29°$, and we get $ϟ\,61°$.

Finally; $ϟ\,60°$ is found from $ϟ\,30°$ by (17).

10. If we combine the use of (14) with the former of equations (16), we can fill the whole table by starting from the limit $x = 30°$: and although errors might accumulate in so long a process, equations (15), (16) give us such easy modes of verification, that this perhaps is not to be feared. From $x = 0$ to $x = 30°$, $ϟx$ may be found with two or three decimal figures more than are wanted in the higher parts of the table, which will obviate this difficulty.

To give conciseness to the following explanation, write y for x in the former of equations (16), then we have

$$(a)\quad \begin{cases} \frac{1}{2}ϟ(2x) = (\frac{1}{2}\pi - x)\,l2 + ϟx - ϟ(90° - x), \\ \frac{1}{3}ϟ(3y) = -\,2yl2 + ϟy + ϟ(60° + y) - ϟ(60° - y). \end{cases}$$

Suppose $ϟx$ to have been found as high as $x = 30°$. Find

$$ϟ\,60° \text{ and } ϟ\,45° \text{ by (17).}$$

Put $x = 1°, 2°,..15°$, and find $ϟx'$ from $x' = 89°$ to $x' = 75°$. In equations (a) make $x = 18°$, $y = 24°$;

$$\therefore\quad \left.\begin{aligned} \tfrac{1}{2}ϟ\,36° + ϟ\,72° &= \tfrac{2}{5}ϟ\pi + ϟ\,18° \\ ϟ\,36° + \tfrac{1}{3}ϟ\,72° &= ϟ\,24° + ϟ\,84° - \tfrac{4}{15}ϟ\pi \end{aligned}\right\}.$$

Since the right-hand members of these two equations are known, we can solve for $ϟ\,36°$ and $ϟ\,72°$.

Put $y = 29°, 28°,....25°$; then $ϟy$, $ϟ(3y)$ and $ϟ(60° + y)$ being known, we can deduce $ϟ(60° - y)$; *i.e.* we find $ϟy'$ from $y' = 31°$ to $y' = 35°$.

Make $y=20^\circ$, and we find $\text{ϟ}40^\circ$: make $x=36^\circ$, and we find $\text{ϟ}54^\circ$.
.... $x=27^\circ$, ... $\text{ϟ}63^\circ$: .... $y=21^\circ$, ... $\text{ϟ}39^\circ$.
.... $y=18^\circ$, ... $\text{ϟ}42^\circ$: .... $x=21^\circ$, ... $\text{ϟ}69^\circ$.
.... $y=23^\circ$, ... $\text{ϟ}37^\circ$: .... $y=\ 9^\circ$, ... $\text{ϟ}51^\circ$.
.... $y=17^\circ$, ... $\text{ϟ}43^\circ$: .... $y=12^\circ$, ... $\text{ϟ}48^\circ$.
.... $y=16^\circ$, ... $\text{ϟ}44^\circ$: .... $x=24^\circ$, ... $\text{ϟ}66^\circ$.
.... $y=22^\circ$, ... $\text{ϟ}38^\circ$: .... $x=33^\circ$, ... $\text{ϟ}57^\circ$.
.... $y=19^\circ$, ... $\text{ϟ}41^\circ$.

Thus the table is filled as high as $x = 45^\circ$; and the gaps in the upper portion of it may be completed by the former method.

11. To expand $\text{ϟ}x$ in converging series, when x does not exceed 30°.

First, put $\sin x = y$,

$$\therefore\quad \text{ϟ}x = -x \log y + \int \sin^{-1} y \,.\, y^{-1} dy.$$

Expand $\sin^{-1} y$ and integrate. There results

$$\text{ϟ}x = -x \log x + 1^{-2} \sin x + \tfrac{1}{2} \,.\, 3^{-2} \sin^3 x + \frac{1.3}{2.4} \,.\, 5^{-2} \sin^5 x + \&c. \ldots (18).$$

Thus, in particular, if $x = \frac{1}{6}\pi$,

$$\text{ϟ}30^\circ = \tfrac{1}{6}\text{ϟ}\pi + 1^{-2} \,.\, 2^{-1} + \tfrac{1}{2} \,.\, 3^{-2} \,.\, 2^{-3} + \frac{1.3}{2.4} 5^{-2} \,.\, 2^{-5} + \&c. \ldots$$

Next, let $S_n = 1^{-n} + 2^{-n} + 3^{-n} + \&c. \ldots$ a known sum; and $x = \pi\omega$;

$$\therefore\quad \log \sin (\pi\omega) = \log (\pi\omega) - S_2 \frac{\omega^2}{1} - S_4 \frac{\omega^4}{2} - S_6 \frac{\omega^6}{3} - \&c. \ldots$$

Integrate:

$$\frac{1}{\pi} \text{ϟ}(\pi\omega) = \omega \{1 - \log \pi\omega\} + S_2 \,.\, \frac{\omega^3}{1.3} + S_4 \,.\, \frac{\omega^5}{2.5} + S_6 \,.\, \frac{\omega^7}{3.7} + \&c. \quad \ldots\ldots (19).$$

To increase the convergence, add to the penultimate series before integration:

$$-\log (1 - \omega^2) = \omega^2 + \tfrac{1}{2}\omega^4 + \tfrac{1}{3}\omega^6 + \ldots \&c.$$

$$\therefore\quad -\log \sin (\pi\omega) + \log (1 - \omega^2)$$

$$= -\log (\pi\omega) + (S_2 - 1) \frac{\omega^2}{1} + (S_4 - 1) \tfrac{1}{2}\omega^4 + (S_6 - 1) \tfrac{1}{3}\omega^6 + \&c.$$

whence

$$\left.\begin{aligned} &\frac{1}{\pi}\text{ϟ}(\pi\omega) = \omega\{3 - \log \pi\omega - \log(1-\omega) - \log(1+\omega)\} - \log \frac{1+\omega}{1-\omega} \\ &\quad + (S_2 - 1) \frac{\omega^3}{1.3} + (S_4 - 1) \frac{\omega^5}{2.5} + (S_6 - 1) \frac{\omega^7}{3.7} + \&c. \ldots \end{aligned}\right\} \ldots (20),$$

and if ω is less than $\frac{1}{6}$, each term of the series is less than 144^{th} of that which precedes it.

If the coefficients are formed into a table, and the series be adapted (if necessary) to the common logarithms, it will enable us to compute $\daleth x$ from $x = 0$ to $x = 30°$ with much ease. The most troublesome part of the calculation, when many decimal places are required, is the multiplying by π, (or by π . hyp. log. 10, as the case may need.)

12. We may modify the process so as to obtain a somewhat simpler series, thus: Since

$$\daleth x = -x \log \sin x + \int_0 x \cot x\, dx,$$

$$\text{also} \quad \cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} - \frac{2x}{9\pi^2 - x^2} - \&\text{c.},$$

$$\text{and} \int_0 \frac{-2x^2\, dx}{r^2\pi^2 - x^2} = 2x - r\pi . \log \frac{r\pi + x}{r\pi - x}; \quad \text{let } x = \pi\omega:$$

$$\left.\begin{aligned} \therefore \ \frac{1}{\pi}\daleth(\pi\omega) &= \omega(1 - l \sin . \pi\omega) \\ &+ \left(2\omega - l . \frac{1+\omega}{1-\omega}\right) + \left(2\omega - 2l\frac{2+\omega}{2-\omega}\right) + \ldots . \\ &\ldots . + \left(2\omega - r \log . \frac{r+\omega}{r-\omega}\right) + \ldots . \end{aligned}\right\} \ldots (21).$$

Take n terms of this series, so that $r = (n-1)$ in the last, and let R equal the remainder. Put N_m for $n^{-m} + (n+1)^{-m} + (n+2)^{-m} + \&\text{c.}$. . and observe that

$$2\omega - r \log \frac{r+\omega}{r-\omega} = -2\omega\left\{\frac{\omega^2}{3r^2} + \frac{\omega^4}{5r^4} + \frac{\omega^6}{7r^6} + \&\text{c.}\right\};$$

$$\therefore \ \tfrac{1}{2}R = -N_2\tfrac{1}{3}\omega^3 - N_4\tfrac{1}{5}\omega^5 - N_6 . \tfrac{1}{7}\omega^7 - \&\text{c.} \ldots (22).$$

If we take the most obvious case of $n = 2$, $N_m = S_m - 1$;

$$\left.\begin{aligned} \therefore \ \frac{1}{\pi}\daleth(\pi\omega) &= 3\omega - \omega \log \sin . \pi\omega - \log\frac{1+\omega}{1-\omega} \\ -(S_2 - 1)\tfrac{1}{3}2\omega^3 &- (S_4 - 1)\tfrac{1}{5}2\omega^5 - (S_6 - 1)\tfrac{1}{7}2\omega^7 - \&\text{c.} \end{aligned}\right\} \ldots (23).$$

If between the two formulas (20) and (23) we eliminate the term which contains $(S_2 - 1)$, we get

$$\frac{3}{\pi}\daleth(\pi\omega) = \omega\{9 - 2\log \pi\omega - \log \sin \pi\omega - 2\log . (1 - \omega^2)\}$$

$$-3\log\frac{1+\omega}{1-\omega} - (S_4 - 1)(1 - \tfrac{1}{2})\tfrac{1}{5}2\omega^5 - (S_6 - 1)(1 - \tfrac{1}{3}) . \tfrac{1}{7}2\omega^7 - \&\text{c.} . .$$

which may have some advantage when $(S_4 - 1) . \frac{1}{5}\omega^5$ is small enough to omit.

13. To expand $\daleth(90^\circ - x)$ when x is small.

$\daleth(\frac{1}{2}\pi - x) = \int \log\cos x\, dx = \daleth\frac{1}{2}\pi + x\log\cos x + \int_0 \tan x\,.\,x dx.$

Let $\tan x = z$, and assume

$$\frac{z\tan^{-1}z}{1+z^2} = A_1 z^2 + A_2 z^4 + A_3 z^6 - \&c.\,;$$

$$\therefore\quad \left.\begin{array}{l}\tan^{-1}z = A_1 z - A_2 z^3 + A_3 z^5 - A_4 z^7 + \&c.\,.\,.\\ \qquad\qquad + A_1 z^3 - A_2 z^5 + A_3 z^7 - \&c.\,.\,.\end{array}\right\}$$

whence $A_1 = 1,\ A_2 = 1 + \frac{1}{3},\ A_3 = 1 + \frac{1}{3} + \frac{1}{7}$, &c. . .

also $\displaystyle\int_0 \tan x\,.\,x dx = \int_0 \frac{z\tan^{-1}z\,.\,dz}{1+z^2} = A_1\tfrac{1}{3}z^3 - A_3\,.\,\tfrac{1}{5}z^5 + A_3\tfrac{1}{7}z^7 - \&c.\,.\,.$

$$= \{\tfrac{1}{3}z^3 - \tfrac{1}{5}z^5 + \tfrac{1}{7}z^7 - \&c.\,.\,.\} - \tfrac{1}{3}\{\tfrac{1}{5}z^5 - \tfrac{1}{7}z^7 + \tfrac{1}{9}z^9 - \&c.\}$$
$$+ \tfrac{1}{5}\{\tfrac{1}{7}z^7 - \tfrac{1}{9}z^9 + \&c.\,.\,.\} - \tfrac{1}{7}\,\&c.\ \&c.\,.\,.\,.\,.$$

Let us henceforth use $\phi_n x$ for $\int_0 \tan^{n-1}x\,.\,dx$,

$$\text{or}\quad \phi_n x = \frac{\tan^n x}{n} - \frac{\tan^{n+2}x}{n+2} + \frac{\tan^{n+4}x}{n+4} - \&c.\,.\,.\,.\,.$$

so that $\phi_1 x = x$; $\phi_2 x = \log\sec x$; and $\phi_{n+2}x = \dfrac{\tan^n x}{n} - \phi_n x$:

and we finally obtain

$$\daleth(\tfrac{1}{2}\pi - x) = \daleth\tfrac{1}{2}\pi + x\log\cos x$$
$$+ \phi_3 x - \tfrac{1}{3}\phi_5 x + \tfrac{1}{5}\phi_7 x - \tfrac{1}{7}\phi_9 x + \&c.\,.\,.\,.\,.(24),$$

When x is $< 10^\circ$, $\phi_7 x$ will not affect the sixth decimal.

To obtain a more converging series, let $v = 1 - \cos x$.

$$\int_0 \tan x\,.\,x dx = \int -x\,.\,d\log\cos x = \int_0 \frac{\cos^{-1}(1-v)\,.\,dv}{1-v}.$$

Now $\cos^{-1}(1-v) = \sqrt{(2v)}\left\{1 + \dfrac{1}{2.3}\,.\,(\tfrac{1}{2}v) + \dfrac{1.3}{2.4.5}\,.\,(\tfrac{1}{2}v)^2 + \&c.\right\}$.

Let, then, $\dfrac{\cos^{-1}(1-v)}{1-v} = \sqrt{(2v)}\,\{B_1 + B_2 v + B_3 v^2 + \&c.\,.\,.\,.\}$

and we get $B_1 = 1$, $B_2 = B_1 + \frac{1}{2}.\frac{1}{3}2^{-1}$; $B_3 = B_2 + \dfrac{1.3}{2.4}\,.\,\tfrac{1}{5}2^{-2}$; &c. . .

whence $B_\infty = 1 + \frac{1}{2}.\frac{1}{3}2^{-1} + \dfrac{1.3}{2.4}\,.\,\tfrac{1}{5}2^{-2} + \dfrac{1.3.5}{2.4.6}\,.\,\tfrac{1}{7}2^{-3} + \&c.\,.\,.$

$$= \frac{1}{\sqrt{2}}\cos^{-1}.\,\{1-1\} = \frac{\pi}{2\sqrt{2}}.$$

To increase therefore the convergence, put $C_n = \dfrac{\pi}{2\sqrt{2}} - B_n$;

so that $$C_{n+1} - C_n = B_n - B_{n+1};$$

$$\therefore \ \frac{\cos^{-1}(1-v)}{1-v} = \frac{\pi}{2\sqrt{2}} \cdot \sqrt{(2v)}\,(1+v+v^2+v^3+\ldots)$$
$$- \sqrt{(2v)}\,(C_1 + C_2 v + C_3 v^2 + C_4 v^3 + \ldots\ldots)$$
$$= \tfrac{1}{2}\pi \cdot \frac{\sqrt{v}}{1-v} - \sqrt{2}\,(C_1 v^{\frac{1}{2}} + C_2 v^{\frac{3}{2}} + C_3 v^{\frac{5}{2}} + \&c.\ldots\ldots),$$

whence $$\int_0 \frac{\cos^{-1}(1-v)}{1-v}\,dv = \tfrac{1}{2}\pi \log \frac{1+\sqrt{v}}{1-\sqrt{v}} - \pi\sqrt{v}$$
$$- 2\sqrt{2}\,\{C_1 \tfrac{1}{3} v^{\frac{3}{2}} + C_2 \tfrac{1}{5} v^{\frac{5}{2}} + \&c.\ldots\ldots\}.$$

Put $C_0 = \dfrac{\pi}{2\sqrt{2}}$ for uniformity;

$$\therefore \ C_1 = C_0 - 1\,;\ C_2 = C_1 - \tfrac{1}{2}\cdot\tfrac{1}{3}2^{-1}\,;\ C_3 = C_2 - \frac{1.3}{2.4}\cdot\tfrac{1}{5}2^{-2}\,;\ \&c.\ldots$$

and C_0, C_1, C_2.. may be easily tabulated. Finally, observing that $\sqrt{(2v)} = 2\sin\tfrac{1}{2}x$, and modifying the fraction under *log*

$$\text{ϟ}(\tfrac{1}{2}\pi - x) = \text{ϟ}\tfrac{1}{2}\pi + x\log\cos x + \tfrac{1}{2}\pi \log \frac{\tan\frac{1}{4}(\frac{1}{2}\pi + x)}{\tan\frac{1}{4}(\frac{1}{2}\pi - x)}$$
$$- 4\sin\tfrac{1}{2}x\,\{C_0 + C_1\cdot\tfrac{1}{3}v + C_2\cdot\tfrac{1}{5}v^2 + C_3\cdot\tfrac{1}{7}v^3 + \&c.\ldots\}\ldots(25)$$

which converges well even when x is as large as 60°.

14. In constructing a table of $\text{ϟ}x$ we need to find $\Delta\text{ϟ}x$, or $\text{ϟ}(x+h) - \text{ϟ}x$.

Taylor's theorem may of course be used, but the law of the terms is cumbrous. As a substitute for it, let us recal equation (2), which gave

$$\sin a \cdot \int_0 \frac{\log x\,dx}{X} = \text{ϟ}(\omega + a) - \text{ϟ}\omega - \text{ϟ}a, \text{ where } \tan\omega = \frac{x\sin a}{1 - x\cos a}.$$

whence

$$\omega = \tan^{-1}\left(\frac{x\sin a}{1 - x\cos a}\right) = x\sin a + \tfrac{1}{2}x^2\sin 2a + \tfrac{1}{3}x^3 \sin 3a + \ldots$$

$$\text{Also } \sin a \cdot \int_0 \frac{\log x\,dx}{X} = \int_0 \log x \cdot d\omega = \omega\log x - \int_0 \omega \cdot \frac{dx}{x};$$

$$\therefore \text{ϟ}(\omega + a) - \text{ϟ}a = \text{ϟ}\omega + \omega\log x$$
$$- 1^{-2}x\sin a - 2^{-2}x^2\sin 2a - 3^{-2}x^3\sin 3a - \&c.$$

To conform to the usual notation, write x, h for a, ω, and then we must put y for x. We hereby find that

$$\text{If } y \text{ stands for } \frac{\sin h}{\sin(x+h)},$$

$$\therefore \ \Delta\text{ϟ}x = \text{ϟ}h + h\log y$$
$$- 1^{-2}y\sin x - 2^{-2}y^2\sin 2x - 3^{-2}y^3\sin 3x - \&c.\ldots(26).$$

If x is large compared with h, and $\daleth h$ is known, y will be small enough to give a good convergence, and the law of the series is simple and convenient. Nevertheless, methods of interpolation, such as the following, will be often better.

First, let $m_1, m_2, m_3 ..$ satisfy the equation

$$x \div \log(1+x) = 1 + m_1 x + m_2 x^2 + m_3 x^3 + \ldots.$$

then $\quad \Delta \int Fx\,dx = h\,\{Fx + m_1 \Delta Fx + m_2 \Delta^2 Fx + \&c.. \ldots\}$

and by slightly modifying the process, it is easy to shew that we have also

$$\Delta \int Fx\,dx = h\,\{F(x+h) - m_1 \Delta Fx + m_2 \Delta^2 F(x-h) - m_3 \Delta^3 F(x-2h) + \&c.\}.$$

where F is any function whatever. Here we assume

$$Fx = -\log \sin x:$$

observe that $\qquad m_1 = \frac{1}{2},\ m_2 = -\frac{1}{12},\ m_3 = \frac{1}{24};$

then, nearly,

$$\Delta \daleth x = -h\,\{l \sin x + \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin x + \tfrac{1}{24}\Delta^3 l \sin x\} \ldots (27),$$

$$\text{or } \Delta \daleth x = -h\,\{l \sin(x+h) - \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin(x-h) - \tfrac{1}{24}\Delta^3 l \sin(x-2h)\} \quad .(28).$$

Take the sum:

$$\therefore\ 2\Delta \daleth x = -h\,[l \sin(x+h) - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l \,.\, \sin(x-h)\} + l \sin x + \tfrac{1}{24}\{\Delta^3 l \sin x - \Delta^3 l \sin(x-2h)\}];$$

or, when the last term is negligible,

$$\Delta \daleth x = -\tfrac{1}{2}h\,[l \sin(x+h) + l \sin x - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l \sin .(x-h)\}] \ldots (29).$$

But perhaps certain series of Legendre's are better still.

Let $M_1, M_2, M_3 \ldots$ be determined by the equation

$$\tfrac{1}{2}x \div \sin^{-1}(\tfrac{1}{2}x) = 1 + M_1 x^2 + M_2 x^4 + M_3 x^6 + \ldots.$$

then $\quad M_1 = \dfrac{1}{24},\quad M_2 = -\dfrac{17}{24^2.\,10},\quad M_3 = \dfrac{367}{8^2.5.6.7.8.9};$

and we have

$$\Delta \int Fx\,dx = h\,\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x-\tfrac{1}{2}h) + M_2 \Delta^4 F(x-\tfrac{1}{2}3h) + \&c\ldots\}$$

$$\text{also} \quad = h\,\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x+\tfrac{1}{2}3h) + M_2 \Delta^4 F(x+\tfrac{1}{2}5h) + \&c\ldots\}$$

which here give

$$\left.\begin{aligned} \Delta \daleth x &= -h\,\{l \sin(x+\tfrac{1}{2}h) + \tfrac{1}{24}\Delta^2 l \sin(x-\tfrac{1}{2}h) - \tfrac{17}{5760}\Delta^4 l \sin(x-\tfrac{1}{2}3h) + \&c.\} \\ \text{also } &= -h\,\{l \sin(x+\tfrac{1}{2}h) + \tfrac{1}{24}\Delta^2 l \sin(x+\tfrac{1}{2}3h) - \tfrac{17}{5760}\Delta^4 l \sin(x+\tfrac{1}{2}5h) + \&c.\} \end{aligned}\right\} \ldots (30),$$

which are easy to us, because we have tables of *log sin*.

Again, let $N_1, N_2, N_3 \ldots$ be such that

$$1 - N_1x^2 + N_2x^4 - N_3x^6 + \&c.\ldots = (1 + M_1x^2 + M_2x^4 + \&c.\ldots)^2;$$

$$\therefore \Delta^2 \int Fx dx = h^2 \{F'(x+h) + N_1\Delta^2 F'x + N_2\Delta^4 F'(x-h) + \&c.\}$$

$$\text{also} = h^2 \{F'(x + h + N_1\Delta^2 F'(x+2h) + N_2\Delta^4 F'(x+3h) + \&c.\},$$

$$\text{where } N_1 = \frac{1}{12}, \quad N_2 = -\frac{1}{240}, \quad N_3 = \frac{31}{4.5.6.7.8.9}.$$

Put $Fx = -\log \sin x$, $F'x = -\cot x$.

$$\left.\begin{aligned} \therefore \Delta^2 \daleth x &= -h^2\{\cot(x+h) \\ &\quad + \tfrac{1}{12}\Delta^2 \cot x - \tfrac{1}{240}\Delta^4 \cot(x-h) + \&c.\} \\ \text{also} \quad &= -h^2\{\cot(x+h) \\ &\quad + \tfrac{1}{12}\Delta^2 \cot(x+2h) - \tfrac{1}{240}\Delta^4 \cot(x+3h) + \&c.\} \end{aligned}\right\} \ldots (31).$$

§. IV.—*Applications of* $\daleth$.

15. (1) To find $\Theta = -\int_0 \log(\sin^2\theta - \sin^2 a)\, d\theta$,

$$\text{or} = -\int_0 \log(\cos^2 a - \cos^2\theta)\, d\theta.$$

Observe that $\sin^2\theta - \sin^2 a = \sin(\theta + a) . \sin(\theta - a)$.

$$\therefore \quad \Theta = \daleth(\theta + a) + \daleth(\theta - a), \quad \text{or} = \daleth(a + \theta) - \daleth(a - \theta).$$

(2) To find $\Theta = -\int_0 \log(\cos\theta - \cos a)\, d\theta$.

Since $\cos\theta - \cos a = 2(\cos^2 \tfrac{1}{2}\theta - \cos^2 \tfrac{1}{2}a)$,

$$\Theta = -\theta \log 2 + 2\daleth\left(\frac{a+\theta}{2}\right) - 2\daleth\left(\frac{a-\theta}{2}\right).$$

(3) If $\Theta = \int_0 \log(1 + \sec\mu \cos\theta)\, d\theta$,

since $\log(1 + \sec\mu\cos\theta) = \log\{\cos\mu - \cos(\pi - \theta)\} - \log\cos\mu$,

$$\therefore \quad \Theta = -\theta\log(2\cos\mu) + 2\daleth\left(\frac{\mu + \pi - \theta}{2}\right) - 2\daleth\left(\frac{\mu - \pi + \theta}{2}\right).$$

(4) If $\Theta = \int_0 \log(1 + \sin\mu\cos\theta)\, d\theta$, put $\tan\eta = \dfrac{\sin\theta}{\tan\frac{1}{2}\mu + \cos\theta}$,

and we had by equation (6)

$$\tfrac{1}{2}\Theta = \daleth(\theta - \eta) + \daleth\eta - \daleth\theta + \theta\log\sin\tfrac{1}{2}\mu - \eta\log\tan\tfrac{1}{2}\mu.$$

Thus $\int \log(a \pm b\cos\theta)\, d\theta$ can always be found by $\daleth$.

(5) If $\Theta = \int_0 \log(\tan\theta + \tan a)\, d\theta$;

$$\text{since } \tan\theta + \tan a = \frac{\sin(\theta + a)}{\cos\theta\cos a},$$

$$\Theta = \daleth\tfrac{1}{2}\pi + \daleth a - \theta\log\cos a - \daleth(\theta + a) - \daleth(\tfrac{1}{2}\pi - \theta).$$

(6) To find $S = 1^{-2}x - 3^{-2}x^3 + 5^{-2}x^5 - 7^{-2}x^7 + \&c.\ldots$. In the process which gave rise to equation (26), put $a = \frac{1}{2}\pi$; $x = \tan\omega$; and observing that $\daleth(\omega + \frac{1}{2}\pi) - \daleth\frac{1}{2}\pi = \daleth\frac{1}{2}\pi - \daleth(\frac{1}{2}\pi - \omega)$, we get

$$S = \omega\log x + \daleth\omega + \daleth(\tfrac{1}{2}\pi - \omega) - \tfrac{1}{2}\daleth\pi.$$

The series S received from Spence a special discussion.

(7) If $\Theta = \int\log(1 - 2n\cos a\tan\theta + n^2\tan^2\theta)\,d\theta$, we reduce this to No. (4), as in the process for finding X. See equation (5).

(8) If $\Theta = \int_0\log(\tan^2\theta + \tan^2\beta)\,d\theta$, make $n = \cot\beta$;

$$\therefore\ \Theta = 2\theta\log\tan\beta + \int_0\log(1 + n^2\tan^2\theta)\,d\theta.$$

The last falls under Ex. (7), as a particular case, when $a = \frac{1}{2}\pi$.

(9) If $\Omega = \int_0\log(1 - 2r\cos a\sin\omega + r^2\sin^2\omega)\,d\omega$, this also may be reduced to $\daleth$, by the following process.

Suppose r positive, $x = \tan\frac{1}{2}\omega$, or $\sin\omega = \dfrac{2x}{1+x^2}$: and $1 + x^2 = \sec^2\frac{1}{2}\omega$;

$$\therefore \Omega = \int_0\log\{1 - 4rx\cos a + (4r^2+2)x^2 - 4rx^3\cos a + x^4\}\,d\omega + 8\daleth\left(\frac{\pi - \omega}{2}\right).$$

Assume the quantity under *log* to be

$$= (1 - 2nx\cos\gamma + n^2x^2)(1 - 2n^{-1}x\cos\gamma + n^{-2}x^2);$$

$\therefore (n + n^{-1})\cos\gamma = 2r\cos a$, and $4r^2 + 2 = 4\cos^2\gamma + n^2 + n^{-2}$.

Let $\tan\nu = n$, $\tan\rho = r$; $\therefore 4r^2 + 4 = 4\cos^2\gamma + (n + n^{-1})^2$;

or $\cos\gamma = r\cos a\sin 2\nu$, and $2r\operatorname{cosec}2\rho = \cos^2\gamma + \operatorname{cosec}^2 2\nu$.

Eliminate γ, and solve for $\operatorname{cosec}2\nu$. The result is, that if we take $\sin 2\zeta = \cos a\sin 2\rho$, and select that root of ζ which makes $\pm\sin\zeta$ least, we have

$$\sin 2\nu = \frac{\cos\rho}{\cos\zeta}, \text{ and } \cos\gamma = \frac{\sin\zeta}{\cos\rho}.$$

Hence, having found ν and γ,

$$\left.\begin{aligned} \text{let}\quad \Omega' &= \int_0\log(1 - 2n\cos\gamma\tan\tfrac{1}{2}\omega + n^2\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \\ \Omega'' &= \int_0\log(1 - 2n^{-1}\cos\gamma\tan\tfrac{1}{2}\omega + n^{-2}\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \end{aligned}\right\}$$

where we pass from Ω' to Ω'' by changing ν to $(\frac{1}{2}\pi - \nu)$; then Ω becomes $= 2\Omega' + 2\Omega'' + 8\daleth\left(\dfrac{\pi - \omega}{2}\right)$.

As in the process of Art. (2), make $\tan\beta = \tan 2\nu \cos\gamma$, $\cos\mu = \sin 2\nu \sin\gamma$, and $\theta = \omega + \beta$; observing that γ and $\frac{1}{2}\omega$ replace a and ω. When ν becomes $(\frac{1}{2}\pi - \nu)$, β becomes $-\beta$ and μ is unchanged. Let $\theta' = \omega - \beta$; then

$$\Omega' = \tfrac{1}{2}\int \log(1 + \sin\mu\cos\theta)\,d\theta - \tfrac{1}{2}\omega\log(2\cos^2\nu) - 2ʖ\left(\frac{\pi-\omega}{2}\right):$$

whence $$\Omega = \int\log(1 + \sin\mu\cos\theta)\,d\theta + \int\log(1 + \sin\mu\cos\theta')\,d\theta' - 2\omega\log\sin 2\nu \quad . . (32),$$

which reduces the case to No. (4).

(10) When F is rational, $\int\log Fx \, . \, \frac{dx}{\sqrt{(p^2 - x^2)}}$ is reducible to the sum or difference of integrals such as

$$\int\log(a + bx)\frac{dx}{\sqrt{(p^2 - x^2)}} \text{ and } \int\log(1 - 2x\cos a + x^2)\frac{dx}{\sqrt{(r^2 - x^2)}}.$$

The former falls under No. (3) or No. (4), if $x = p\cos\theta$. The latter is the case of No. (9), if $x = r\cos\omega$. Thus $\int\log Fx \, . \, \frac{dx}{\sqrt{(p^2 - x^2)}}$ is wholly reducible to ʖ.

(11) To find $\int\frac{x^m \log x\, dx}{(a + bx^2)^{n+\frac{1}{2}}}$, when m and n are integers, positive or negative.

Represent the integral by $V_{m,n}$ and let $U_{m,n} = \frac{x^m \log x}{(a + bx^2)^{n-\frac{1}{2}}}$. Differentiate $U_{m,n}$ and integrate back again; and we obtain

$$U_{m,n} - \int\frac{x^{m-1}\,dx}{(a + bx^2)^{n-\frac{1}{2}}} = mV_{m-1,n-1} - (2n-1)\,bV_{m+1,n} \quad . . . . . (1),^*$$

$$= maV_{m-1,n} - (2n - m - 1)\,bV_{m+1,n} \quad . . (2),^*$$

$$= (2n-1)aV_{m-1,n} - (2n-m-1)\,bV_{m-1,n-1} \quad . . (3).^*$$

When $m = 0$, the first gives $V_{1,n}$ but fails to reduce $V_{-1,n-1}$ to $V_{1,n}$: also the second then merges in the first. Yet by the third, $V_{-1,n-1}$ can be reduced to $V_{-1,n}$. When $m = 2n - 1$, the second and third coincide, and give $V_{2n-2,n}$, but fail to reduce $V_{2n-2,n-1}$ to $V_{2n-2,n}$.

There being no other cases of failure, and $V_{2n-2,n-1}$ being reducible by the first formula to $V_{\pm4,\pm2}$, $V_{\pm2,\pm1}$ and finally to $V_{0,0}$; it is evident that $V_{m,n}$ is universally reducible to one of the three, $V_{1,0}$, $V_{0,0}$, $V_{-1,0}$, of which the first (being included under $V_{1,n}$) can be found by common methods. This is the case of m being positive and odd. If m is negative and odd, $V_{m,n}$ is reduced to $V_{-1,0}$; if m is even, it is reduced to $V_{0,0}$.

In $V_{-1,0}$ let $x = y^{-1}$; $\therefore \int \frac{\log x\, dx}{x\sqrt{(a+bx^2)}} = \int \frac{\log y\, dy}{\sqrt{(ay^2+b)}}$; which is of the same form as $V_{0,0}$; which alone remains. When $a = 1$ and $b = -1$, put $x = \sin \omega$, $\therefore\ V_{0\cdot 0} = -\text{ϟ}\omega$. When $a = -1$ and $b = 1$, let $y = x + \sqrt{(x^2-1)}$, $\frac{dx}{\sqrt{(x^2-1)}} = \frac{dy}{y}$; $\log x = \log (1+y^2) - \log (2y)$;

$$\therefore\quad V_{0,0} = \int \log (1+y^2)\, \frac{dy}{y} - \int \log (2y)\, \frac{dy}{y}$$

$$= \tfrac{1}{2} L\,(1+y^2) - \tfrac{1}{2}\log^2 (2y) + \text{const.}$$

When $\quad a = 1$ and $b = 1$, let $y = x + \sqrt{(x^2+1)}$:

$$\therefore\quad V_{0.0} = \log y \,.\, \log (y^2-1) - \tfrac{1}{2} L\,(y^2) - \tfrac{1}{2}\log^2 (2y) + \text{const.}$$

To recapitulate then: $V_{m,n}$ is always reducible to ϟ or L; and in particular, $\int \frac{x^{2m-1} \log x\, dx}{(a+bx^2)^{n+\frac{1}{2}}}$ can be found by circular arcs and logarithms;

$$\int \frac{x^{\pm 2m} \log x\, dx}{(1-x^2)^{n+\frac{1}{2}}} \text{ is reducible to ϟ; } \int \frac{x^{\pm 2m} \log x\, dx}{(x^2 \pm 1)^{n+\frac{1}{2}}} \text{ to } L;$$

$$\int \frac{x^{-2m+1} \log x\, dx}{(x^2-1)^{n+\frac{1}{2}}} \text{ to ϟ; } \int \frac{x^{-2m+1} \log x\, dx}{(1 \pm x^2)^{n+\frac{1}{2}}} \text{ to } L;$$

where m and n are integers, m positive and n either positive or negative.

§. V.—*On the Higher Transcendents derivable from* ϟ.

16. Spence has imagined the integrals L^3, L^4, L^5. . deduced from L^2 or L, by the law $L^n(1+x) = \int L^{n-1}(1+x)\, x^{-1}\, dx$; and has exhibited various fundamental properties of L^n. Put

$$x = e^{2\omega}, \quad \therefore\ e^{2\omega} + 1 = (e^{\omega} + e^{-\omega})\, e^{\omega},$$

$$\text{or } \log (x+1) = \log (\epsilon^{\omega} + \epsilon^{-\omega}) + \omega, \quad \text{and } x^{-1} dx = 2d\omega;$$

$$\therefore\ L\,(1+x) = \int \log (\epsilon^{\omega} + \epsilon^{-\omega})\, 2d\omega + \omega^2.$$

When ω changes to $\omega\sqrt{(-1)}$, the integral here becomes $2\sqrt{(-1)}\,.\int \log (2 \cos \omega)\, d\omega$, which exhibits the relation which exists between L and ϟ by imaginaries.

17. Since $d\omega \propto x^{-1}dx$, we may imagine a series of functions $\text{ϟ}^3, \text{ϟ}^4, \text{ϟ}^5$. . analogous to L^3, L^4, L^5. ., by the law $\text{ϟ}^3 x = \int_0 \text{ϟ} x\, dx$, and generally $\text{ϟ}^n x = \int_0 \text{ϟ}^{n-1} x\, dx$; and we now regard ϟ as virtually ϟ^2. Write also

$$\lambda_2 x = -\int_0 \log x\, dx, \quad \lambda_n x = \int_0 \lambda_{n-1}\, x\, dx;$$

then $\lambda_2 x = x(1 - \log x)$,

$$\lambda_{n+1} x = \frac{x^n}{1.2\ldots n}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n} - \log x\right\} \ldots (33).$$

Also, as $ϟ^2 x = \lambda_2 x + H_1 \dfrac{x^3}{1.3} + H_2 \dfrac{x^5}{2.5} + H_3 . \dfrac{x^7}{3.7} + \&c.\ldots\ldots$

if H_n stands for $\pi^{-2n} S_{2n}$; (see equation 19.)

$$\therefore ϟ^n x = \lambda_n x + \frac{2H_1 x^{n+1}}{2.3\ldots(n+1)} + \frac{2H_2 x^{n+3}}{4.5\ldots(n+3)} + \frac{2H_3 x^{n+4}}{6.7\ldots(n+5)} + \&c. \ldots\ldots(34).$$

18. Since $2ϟ^2(\tfrac{1}{2}x) = xl2 + 1^{-3}\sin x + 2^{-2}\sin 2x + 3^{-2}\sin 3x + \&c.\ldots$ perpetual integration, with suitable addition of constants, gives

$$\left.\begin{aligned} 2^{2n-1}. ϟ^{2n}(\tfrac{1}{2}x) &= \frac{x^{2n-1} l2}{1.2\ldots(2n-1)} \\ &+ \frac{x^{2n+3} S_3}{1.2\ldots(2n-3)} - \frac{x^{2n-5} S_5}{1.2\ldots(2n-5)} + \&c.\ldots \pm \frac{x}{1} S_{2n-1} \\ &- 1^{-2n}\sin(x - 2n.\tfrac{1}{2})\pi - 2^{-2n}\sin(2x - 2n\,\tfrac{1}{2}\pi) \\ &\qquad - 3^{-2n}\sin(3x - 2n\,\tfrac{1}{2}\pi) - \&c. \end{aligned}\right\} \ldots (35).$$

$$\left.\begin{aligned} 2^{2n} ϟ^{2n+1}(\tfrac{1}{2}x) &= \frac{x^{2n} l2}{1.2..2n} \\ &+ \frac{x^{2n-2} S_3}{1.2..(2n-2)} - \frac{x^{2n-4} S_5}{1.2..(2n-4)} + \&c.\ldots \mp S_{2n+1} \\ &- 1^{-2n-1}\sin\{x - (2n+1).\tfrac{1}{2}\pi\} \\ &\qquad - 2^{-2n-1}\sin\{2x - (2n+1)\,\tfrac{1}{2}\pi\} - \&c.\ldots \end{aligned}\right\} \ldots (36).$$

And if in these we put $x = 2\pi$, we get

$$ϟ^3 \pi = \frac{\pi^2 l2}{1.2} : \quad ϟ^4 \pi = \frac{\pi^3 l2}{1.2.3} + 2^{-2}.\frac{\pi}{1} S_3 :$$

$$ϟ^5 \pi = \frac{\pi^4 l2}{1.2.3.4} + 2^{-2}.\frac{\pi^2}{1.2}.S_3 :$$

$$ϟ^6 \pi = \frac{\pi^5 l2}{1.2..5} + 2^{-2}.\frac{\pi^3}{1.2.3} S_3 - 2^{-4}.\frac{\pi}{1} S_5 :$$

$$ϟ^7 \pi = \frac{\pi^6 l2}{1.2..6} + 2^{-2}.\frac{\pi^4}{1.2.3.4} S_3 - 2^{-4}.\frac{\pi^2}{1.2}.S_5 :$$

The law is evident. After the two first terms, the signs are alternate. Thus $ϟ^n \pi$ is known.

19. If in equations (35), (36) we make $x = \pi$; and with reference to the latter, observe that

$$1^{-m} - 2^{-m} + 3^{-m} - 4^{-m} + \&c.. . = (1 - 2^{-m+1}) S_m;$$

we obtain

$$2^{2n-1} \digamma^{2n}(\tfrac{1}{2}\pi) = \frac{\pi^{2n-1} l2}{1.2..(2n-1)} + \frac{\pi^{2n-3} S_3}{1.2..(2n-3)} - \frac{\pi^{2n-5} S_5}{1.2..(2n-5)} + \ldots \pm \frac{\pi}{1} S_{2n-1} \ldots (37):$$

$$2^{2n} \digamma^{2n+1}(\tfrac{1}{2}\pi) = \frac{\pi^{2n} l2}{1.2..2n} + \frac{\pi^{2n-2} S_3}{1.2..(2n-2)} - \&c.\ldots \mp (2 - 2^{-2n}) S_{2n+1} \ldots (38):$$

by which $\digamma^n(\frac{1}{2}\pi)$ is known.

20. If we perpetually integrate $\digamma^2 x + \digamma^2(\pi - x) = \digamma^2 \pi$, we get

$$\digamma^n x + (-1)^n . \digamma^n(\pi - x) = \frac{x^{n-2} \digamma^2 \pi}{1.2..(n-2)} - \frac{x^{n-3} \digamma^3 \pi}{1.2..(n-3)} + \ldots + (-1)^n . \digamma^n \pi \ldots (39),$$

which reduces $\digamma^n x$ to $\digamma^n(\pi - x)$.

21. Perpetually integrate

$$\tfrac{1}{2} \digamma^2(2x) = \digamma^2 x - \digamma^2(\tfrac{1}{2}\pi - x) + \digamma^2(\tfrac{1}{2}\pi) - x l2;$$

$$\therefore 2^{-n+1} \digamma^n(2x) = \digamma^n x + (-1)^{n-1} \digamma^n(\tfrac{1}{2}\pi - x) - \frac{x^{n-1} l2}{1.2..(n-1)} + \frac{x^{n-2} \digamma^2 \frac{1}{2}\pi}{1.2..(n-2)} - \frac{x^{n-3} \digamma^3 \frac{1}{2}\pi}{1.2..(n-3)} .. \pm \digamma^n \tfrac{1}{2}\pi ..(40),$$

which may be used exactly as equation (14) in Art. 9.

Make $x = \frac{1}{2}\pi$, and multiply by 2^{n-1};

$$\therefore \digamma^n \pi = 2^{n-1}\{1 + (-1)^{n-1}\} \digamma^n \tfrac{1}{2}\pi - \frac{\pi^{n-1} l2}{1.2..(n-1)} + \frac{2\pi^{n-2} \digamma^2 \frac{1}{2}\pi}{1.2..(n-2)} - \frac{2^2 . \pi^{n-3} \digamma^3 \frac{1}{2}\pi}{1.2..(n-3)} + \ldots \pm 2^{n-1} \digamma^n \tfrac{1}{2}\pi \ldots (41).$$

22. To complete the view of $\digamma^n x$, we ought to embrace the cases of $x < 0$ and $x > \pi$.

It is obvious that

$$\digamma^{2n}(-x) = -\digamma^{2n} x; \text{ but } \digamma^{2n+1}(-x) = \digamma^{2n+1} x \ldots (42).$$

Also, by perpetual integration of $\digamma^2(n\pi + x) = \digamma^2(n\pi) + \digamma^2 x$,

$$\digamma^m(n\pi + x) = \digamma^m(n\pi) + \frac{x}{1} \digamma^{m-1}(n\pi) + \frac{x^2}{1.2} \digamma^{m-2}(n\pi) + \ldots + \frac{x^{m-2}}{1.2 \ldots (m-2)} . \digamma^2(m\pi) + \digamma^m x \ldots (43).$$

But we farther want to express $\varsigma^m(n\pi)$ by means of $\varsigma^m\pi$, $\varsigma^{m-1}\pi, \ldots \varsigma^2\pi$, for which we begin with ς^3 and proceed to ς^4, ς^5 in succession.

For $\varsigma^3(n\pi)$, let $x = \pi$, and $n = 1, 2, 3, 4 \ldots.$

$$\therefore\ \varsigma^3(2\pi) = \varsigma^3(\pi + \pi) = \varsigma^3\pi + \frac{\pi}{1}\varsigma^2\pi + \varsigma^3\pi = 2\varsigma^3\pi + \frac{\pi}{1}\varsigma^2\pi,$$

$$\varsigma^3(3\pi) = \varsigma^3(2\pi + \pi) = \varsigma^3(2\pi) + \frac{\pi}{1}\varsigma^2(2\pi) + \varsigma^3\pi$$

$$= 3\varsigma^3\pi + 3\frac{\pi}{1}\varsigma^2\pi,$$

$$\varsigma^3(4\pi) = \varsigma^3(3\pi) + \frac{\pi}{1}\varsigma^2(3\pi) + \varsigma^3\pi = 4\varsigma^3\pi + 6\frac{\pi}{1}\varsigma^2\pi.$$

Generally, $\varsigma^3(n\pi) = n\varsigma^3\pi + n\,.\,\frac{n-1}{2}\,.\,\frac{\pi}{1}\varsigma^2\pi.$

For $\varsigma^4(n\pi)$ proceed by the same steps:

$$\varsigma^4(2\pi) = 2\varsigma^4\pi + \frac{\pi}{1}\varsigma^3\pi + \frac{\pi^2}{1.2}\varsigma^2\pi:$$

$$\varsigma^4(3\pi) = 3\varsigma^4\pi + 3\frac{\pi}{1}\varsigma^3\pi + 5\frac{\pi^2}{1.2}\varsigma^2\pi.$$

Generally, $\varsigma^4(n\pi) = n\varsigma^4\pi + \Sigma n\,.\,\frac{\pi}{1}\varsigma^3\pi + \Sigma n^2\,.\,\frac{\pi^2}{1.2}\,.\,\varsigma^2\pi.$

It is easy to see that we may assume

$$\varsigma^m(n\pi) = n\varsigma^m\pi + \psi_1 n\,.\,\frac{\pi}{1}\varsigma^{m-1}\pi + \psi_2 n\,.\,\frac{\pi^2}{1.2}\varsigma^{m-2}\pi + \ldots$$

to $(m-1)$ terms,

and that the functions ψ do not contain (m). To determine their law, put $x = \pi$ in equation (43), so as to obtain $\varsigma^m\{(n+1)\,.\,\pi\}$. Also write $(n+1)$ for n in the assumed series, and compare the results. This gives

$$\Delta\psi_1 n = n, \qquad \Delta\psi_2 n = n + 2\psi_1 n,$$

$$\Delta\psi_3 n = n + 3\psi_1 n + 3\psi_2 n, \qquad \Delta\psi_4 n = n + 4\psi_1 n + 6\psi_2 n + 4\psi_3 n,$$

$$\Delta\psi_5 n = n + 5\psi_1 n + 10\psi_2 n + 10\psi_3 n + 5\psi_4 n,$$

where the law is obvious.

Write N_r for $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\ldots\frac{n-r+1}{r}$, and integrate the above;

$$\therefore\ \psi_1 n = \Sigma n = N_2, \quad \psi_2 n = N_2 + 2N_3,$$

$$\psi_3 n = N_2 + 3N_3 + 3\{N_3 + 2N_4\}$$

$$= N_2 + 6N_3 + 6N_4,$$

$$\psi_4 n = N_2 + 4N_3 + 6\{N_3 + 2N_4\} + 4\{N_3 + 6N_4 + 6N_5\}$$
$$= N_2 + 14N_3 + 36N_4 + 24N_5.$$

Generally, it is easy to satisfy ourselves that

$$\psi_r n = \Delta 0^r . N_2 + \Delta^2 0^r . N_3 + \Delta^3 0^r . N_4 + \ldots \text{ to } r \text{ terms};$$
$$\therefore \Delta\psi_r n = \Delta 0^r . N_1 + \Delta^2 0^r . N_2 + \Delta^3 0^r . N_3 + \ldots \text{ to } r \text{ terms},$$
$$= n^r. \qquad \text{Hence } \psi_r n = \Sigma n^r.$$

Finally then we obtain

$$ℎ^m (n\pi) = n ℎ^m \pi + \Sigma n . \frac{\pi}{1} ℎ^{m-1}\pi + \Sigma n^2 . \frac{\pi^2}{1.2} . ℎ^{m-2}\pi + \ldots$$
$$\ldots \text{ to } (m-1) \text{ terms} \ldots (44),$$

where $\Sigma n^r = 0^r + 1^r + 2^r + \ldots (n-1)^r.$

[*To be continued.*]

ON THE LAWS OF EQUILIBRIUM AND MOTION OF SOLID AND FLUID BODIES.

By Samuel Haughton.

(*Continued from* Vol. I. p. 173.)

The differential equations of motion of solid bodies are deduced from (11), by writing $X - \frac{d^2\xi}{dt^2}$, $Y - \frac{d^2\eta}{dt^2}$, $Z - \frac{d^2\zeta}{dt^2}$, for X, Y, Z, and consequently are the following:

$$\epsilon \frac{d^2\xi}{dt^2} = \epsilon X + P, \quad \epsilon \frac{d^2\eta}{dt^2} = \epsilon Y + Q, \quad \epsilon \frac{d^2\zeta}{dt^2} = \epsilon Z + R \ldots (12).$$

Let us suppose that no external forces of any kind act upon the body, and endeavour to satisfy the equations of motion by the particular integral for plane waves,

$$\xi = \cos a . f(\omega), \quad \eta = \cos \beta . f(\omega), \quad \zeta = \cos \gamma . f(\omega),$$
$$\omega = lx + my + nz - vt.$$

Substituting these values of ξ, η, ζ in the differential equations

$$\epsilon \frac{d^2\xi}{dt^2} = P, \quad \epsilon \frac{d^2\eta}{dt^2} = Q, \quad \epsilon \frac{d^2\zeta}{dt^2} = R,$$

we shall obtain the following equations of condition among the constants,

$$\left.\begin{aligned} \epsilon v^2 . \cos a &= p' \cos a + h' \cos \beta + g' \cos \gamma, \\ \epsilon v^2 . \cos \beta &= q' \cos \beta + f' \cos \gamma + h' \cos a, \\ \epsilon v^2 . \cos \gamma &= r' \cos \gamma + g' \cos a + f' \cos \beta, \end{aligned}\right\} \ldots (13),$$

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.
PART II.

By FRANCIS W. NEWMAN.

(*Continued from* p. 100).

On the Integrals $\left.\begin{aligned}\Lambda(x, a) &= \tfrac{1}{2}\int_0 \frac{\log x\,(x - \cos a)\,dx}{x^2 - 2x\cos a + 1}\\ \lambda(x, a) &= \tfrac{1}{2}\int_0 \log(x^2 - 2x\cos a + 1)\,\frac{dx}{x}\end{aligned}\right\}$

which are related by the equation $\Lambda(x, a) + \lambda(x, a) = \tfrac{1}{2}\log x \log X$, *if* X *stands for* $(x^2 - 2x\cos a + 1)$.

§ I.—*Simplest cases of* Λ.

1. It was shewn in Part I. that all the integrals included in $\int F_1 x \log F_2 x\, dx$ are reducible to common forms in conjunction with three peculiar integrals, Lx, ⅂x, and $\Lambda(x, a)$, of which the last alone remains to be treated. We suppose a to be between 0 and π, unless the contrary is stated. We also generally suppose x to be positive. When it comes out negative in any formula, we can restore it by help of the identical equation

$$\Lambda(-x, a) = -\Lambda(x, \pi - a),$$

which subsists by virtue of the convention (already proposed) that $\log x$ is always to mean $\tfrac{1}{2}\log(x^2)$.

Assuming $x = \dfrac{\sin\omega}{\sin(\omega + a)}$, or $\tan\omega = \dfrac{x\sin a}{1 - x\cos a}$,

$$\text{we get } \sqrt{X} = \frac{\sin a}{\sin(\omega + a)};$$

$$\text{whence } \Lambda(x, a) = \int \log\frac{\sin(\omega + a)}{\sin\omega}\,.\,d\log\sin(\omega + a);$$

which we shall hereafter denote by $\chi(\omega, a)$; so that $\Lambda(x, a)$ and $\chi(\omega, a)$ are identical forms. This substitution is chiefly of use in enabling us to understand the nature of other transformations at which we shall arrive. For the present, when ω is named, it is supposed to bear this relation to x.

2. To find the *complete function* $\Lambda(1, a)$, which $= -\lambda(1, a)$.

$$\text{Since } \lambda(x, a) = \tfrac{1}{2}\int_0 \log X\,\frac{dx}{x},\quad \frac{d\lambda}{da} = \int_0 \frac{\sin a}{X}\,dx = \omega.$$

Make $x = 1$, $\therefore \tan\omega = \dfrac{\sin a}{1 - \cos a} = \cot\tfrac{1}{2}a$, or $\omega = \tfrac{1}{2}(\pi - a)$.

Integrate $\frac{d\lambda}{da} = \frac{1}{2}(\pi - a)$; $\therefore \lambda(1, a) = c + \frac{1}{2}\pi a - \frac{1}{4}a^2$.

To find c, make $a = 0$; $\lambda(x, 0) = \int_0 \log(1 - x)\frac{dx}{x} = L(1 - x)$,

$$\text{whence } \lambda(1, 0) = L0, \text{ or } c = -\tfrac{1}{6}\pi^2,$$

$$\left.\begin{array}{l} \therefore\ \Lambda(1, a) = -\lambda(1, a) = \frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{4}a^2. \\ \text{Hence also } \Lambda(1, \pi - a) = \frac{1}{4}a^2 - \frac{1}{12}\pi^2. \end{array}\right\} \ldots\ldots(1).$$

3. To find Λ at special values of a.

At the extreme values, ($a = 0$ and $a = \pi$), X is an algebraic square, $(x \pm 1)^2$. Hence

$$\left.\begin{array}{l} \Lambda(x, 0) = Lx - L0 \\ \Lambda(x, \pi) = L(-x) - L0 = lxl(1 + x) - L(1 + x) \end{array}\right\} \ldots(2).$$

In the following process, we for a moment suppose a to increase from 0 to any magnitude; and, to shew both variables, write $f(x, a)$ instead of X. Then, by a well-known formula of Trigonometry,

$$f(x^n, na) = f(x, a).f\left(x, a + \frac{2\pi}{n}\right).f\left(x, a + \frac{4\pi}{n}\right)\ldots f\left(x, a + \frac{2n-2}{n}\pi\right).$$

Differentiate logarithmically: multiply by $(2n)^{-1}.\log(x^n) = 2^{-1}\log x$, and integrate: then

$$\frac{1}{n}\Lambda(x^n, na) = \Lambda(x, a) + \Lambda\left(x, \ a + \frac{2\pi}{n}\right)$$

$$+ \Lambda\left(x, a + \frac{4\pi}{n}\right) + \ldots\ldots + \Lambda\left(x, a + \frac{2n-2}{n}\pi\right) \ldots\ldots(3).$$

In particular, if $n = 2$, $\cos(a + \pi) = \cos(\pi - a)$,

$$\left.\begin{array}{l} \therefore\ \frac{1}{2}\Lambda(x^2, 2a) = \Lambda(x, a) + \Lambda(x, \pi - a) \\ \qquad\qquad\quad = \Lambda(x, a) + \Lambda(-x, a) \end{array}\right\} \ldots\ldots(4),$$

which has a certain analogy to $Lx + L(-x) = \frac{1}{2}L(x^2) + \frac{3}{2}L0$.

When $a = \frac{\pi}{n}$, we have

$$\left.\begin{array}{l} \frac{1}{n}\Lambda(x^n, \pi) \text{ or } L(-x^n) - L0 \\ = \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots + \Lambda\left(x, \frac{2n-1}{n}\pi\right) \end{array}\right\}.$$

Introvert the terms; and to make every a fall between 0 and π, observe that $\cos\frac{2n - r}{n}\pi = \cos\frac{r\pi}{n}$. Then we find that

$$\left.\begin{array}{l} \text{if } S = \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots\ldots \\ \text{when } n \text{ is even, } S \text{ to } \frac{n}{2} \text{ terms} = \frac{1}{2n}\Lambda(x^n, \pi), \\ \text{and when } n \text{ is odd,} \\ \qquad S \text{ to } \frac{n-1}{2} \text{ terms} = \frac{1}{2n}\Lambda(x^n, \pi) - \frac{1}{2}\Lambda(x, \pi) \end{array}\right\} \ldots (5).$$

In particular,

$$\left.\begin{array}{l} \text{If } n = 2, \quad \Lambda(x, \tfrac{1}{2}\pi) = \tfrac{1}{4}\Lambda(x^2, \pi): \\ \text{If } n = 3, \quad \Lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, \pi) - \tfrac{1}{2}\Lambda(x, \pi) \end{array}\right\} \ldots (6).$$

If in equation (3) we make $na = 2\pi$, and n is odd, we similarly have

$$\Lambda\left(x, \frac{2\pi}{n}\right) + \Lambda\left(x, \frac{4\pi}{n}\right) + \ldots\ldots + \Lambda\left(x, \frac{n-1}{n}\pi\right)$$
$$= \frac{1}{2n}\Lambda(x^n, 0) - \tfrac{1}{2}\Lambda(x, 0) \ldots\ldots (7):$$

If $n = 3$ in the last,

$$\Lambda(x, \tfrac{2}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, 0) - \tfrac{1}{2}\Lambda(x, 0) \ldots\ldots\ldots (8).$$

Thus we know $\Lambda(x, a)$ in finite functions of x, by means of L, when a has any of the values 0, π, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{2}{3}\pi$. It will afterwards appear, that the assertion may be extended to the case of a = any of these, *divided by* 2^n, when n is an arbitrary integer.

4. To find Λ, when x (at the upper limit) is a given function of a.

Generally, if $u = Fx$, $V = \psi(x, a)$, and $U = \int u\,dV$,

$$\frac{dU}{da} = \int u \frac{d^2V}{dx\,da}\,dx = \int u \frac{dV'}{dx}\,dx, \text{ if } V' = \frac{dV}{da};$$

Integrate by parts, and the last is $(uV' - \int V'du)$; and the *total* differential

$$d(U) = \frac{dU}{dx}dx + \frac{dU}{da}da = u\frac{dV}{dx}dx + \left(u\frac{dV}{da} - \int V'du\right)da$$
$$= u.d(V) - (\int V'du)\,da.$$

In the present case, let $u = \frac{1}{2}\log x$, $V = \log X$, and observe that $V' = 2xX^{-1}\sin a$, which vanishes with x; and $\int V'du = \omega$, which also vanishes with x, as $\frac{dU}{da}$ or (here) $\frac{d\Lambda}{da}$ ought to do

No constant then is needed in integrating; and we get

$$d(\Lambda) = \tfrac{1}{2}\log x . d(\log X) - \omega d\alpha \ldots\ldots\ldots (9)$$

for the total differential, when x is a function of α.

The extreme case of $x = (\cos\alpha)^0 = 1$, reproduces equation (1), as it ought.

Assume $x = 2\cos\alpha$, $X = 1$, $\tan\omega = -\tan 2\alpha$, $\omega = \pi - 2\alpha$; $\therefore d\Lambda = -(\pi - 2\alpha)\,d\alpha$. Observe that Λ vanishes when $x = 0$, and consequently (here) when $\alpha = \frac{1}{2}\pi$, and we get

$$\Lambda \text{ or } \Lambda(2\cos\alpha, \alpha) = (\tfrac{1}{2}\pi - \alpha)^2 \ldots\ldots (10).$$

This is the most remarkable equation we have yet met.

It may also be denoted (if $a = \cos\alpha$) by

$$\int_0^{2a} \frac{\log x(x-a)}{x^2 - 2ax + 1} . dx = (\sin^{-1} a)^2.$$

Once more, assume $x = \cos\alpha = a$, $\therefore X = 1 - a^2$, $\tan\omega = \cot\alpha$, $\omega = \frac{1}{2}\pi - \alpha$; whence

$$d\Lambda = \tfrac{1}{2}\log a\, d\log(1 - a^2) - (\tfrac{1}{2}\pi - \alpha)\,d\alpha,$$

$$\therefore \quad \Lambda = \tfrac{1}{4}L(a^2) + \tfrac{1}{2}(\tfrac{1}{2}\pi - \alpha)^2 + \text{const};$$

$$\text{or } \Lambda(\cos\alpha, \alpha) = \tfrac{1}{4}L(\cos^2\alpha) + \tfrac{1}{2}(\tfrac{1}{3}\pi^2 - \pi\alpha + \alpha^2) \ldots . (11):$$

$$= \tfrac{1}{4}L(\cos^2\alpha) + \Lambda(1, \alpha) + \tfrac{1}{4}\alpha^2.$$

The constant is determined as before, by observing that Λ must vanish when $\alpha = \frac{1}{2}\pi$; also $L0 = -\frac{1}{6}\pi^2$.

There is an infinity of other assumptions ($x = F\alpha$) which make (9) integrable in finite terms. Again, ω may be expanded in a series of cosines or sines of multiples of α, after which we may try to integrate. But the cases in which these processes succeed appear to be more easily treated by some of the methods which follow.

§ II.—*On changing* $\Lambda(x, \alpha)$ *to* $\Lambda(y, \alpha)$.

5. As long as x is small (say $x < \pm\frac{1}{2}$), we may develop $\log X$ by the well-known series, and obtain

$$-\lambda(x, \alpha) = \frac{x\cos\alpha}{1^2} + \frac{x^2\cos 2\alpha}{2^2} + \frac{x^3\cos 3\alpha}{3^2} + \&c. \ldots . (12),$$

from which $\lambda(x, \alpha)$ and $\Lambda(x, \alpha)$ are found. But when x is not small enough, we may try to reduce $\Lambda(x, \alpha)$ to $\Lambda(y, \alpha)$, in which y is small.

For the present, let Y stand for $y^2 - 2y\cos\alpha + 1$.

6. Then, *first*, put $xy = 1$, $\therefore\ Y = Xx^{-2}$, $\log y = -\log x$, $d \log Y = d \log X - 2d \log x$;

$$\Lambda x + \Lambda y = \tfrac{1}{2}\int lx\,(dlX - dlY) = \int lx\, d\, lx$$
$$= \tfrac{1}{2}\log^2 x + \text{const};$$
$$\text{or}\quad \Lambda x + \Lambda x^{-1} = 2\Lambda 1 + \tfrac{1}{2}\log^2 x \ldots\ldots\ldots (13),$$

which serves to reduce Λx, when x is > 1, to Λy, where y is < 1.

Cor. 1. When $x = \infty$, Λx converges to $2\Lambda 1 + \frac{1}{2}\log^2 x$.

Cor. 2. As $Lx + Lx^{-1} = \frac{1}{2}\log^2 x$,

$$\therefore (\Lambda x - Lx) + (\Lambda x^{-1} - Lx^{-1}) = 2\Lambda 1.$$

Secondly, suppose $x + y = 2\cos\alpha$, $X = 1 - xy = Y$;

$$\therefore\ \Lambda x + \Lambda y = \tfrac{1}{2}\int (lx + ly)\, dl\,(1 - xy)$$
$$= \tfrac{1}{2}\int l(xy).dl(1 - xy) = \tfrac{1}{2}L(xy) + C:$$

$$\left.\begin{array}{l}\text{whence } \Lambda x + \Lambda y - \Lambda(2\cos\alpha) = \frac{1}{2}L(xy) - \frac{1}{2}L0,\\ \qquad\text{when } x + y = 2\cos\alpha.\end{array}\right\}\ \ldots (14).$$

As we know $\Lambda(2\cos\alpha)$ from equation (10), this enables us to find Λx by means of Λy, whenever x is near to $2\cos\alpha$. It also gives

$$2\Lambda\cos\alpha - \Lambda(2\cos\alpha) = \tfrac{1}{2}L\cos^2\alpha - \tfrac{1}{2}L0,$$

which is verified by (10) and (11).

Make $x = 1$, and we find

$$\Lambda(2\cos\alpha - 1) = \tfrac{1}{2}L(2\cos\alpha - 1) + \tfrac{1}{6}\pi^2 - \tfrac{1}{2}3\pi\alpha + \tfrac{1}{4}5\alpha^2 \ldots (14^*).$$

We may combine the two last integrations, by supposing $x^{-1} + y = 2\cos\alpha$, which gives

$$\Lambda x - \Lambda y + \tfrac{1}{2}\alpha^2 = \tfrac{1}{2}\log^2 x - \tfrac{1}{2}L\frac{y}{x} \ldots\ldots (15).$$

Again, in this write y^{-1} for y, and eliminate Λy^{-1} by means of (13), and $L(x^{-1}y^{-1})$ by means of the known properties of L; then

$$\left\{\begin{array}{l} x^{-1} + y^{-1} = 2\cos\alpha;\\ \Lambda x + \Lambda y - \pi\left(\frac{1}{3}\pi - \alpha\right) = \frac{1}{2}L(xy) + \frac{1}{4}\log^2\left(\dfrac{x}{y}\right)\end{array}\right\}\ \ldots (16).$$

But in this, the arbitrary constant is liable to change by reason of discontinuity, if x or y passes through zero.

7. The four suppositions here made have something in common. In (13), (14), and (16), we find

$$\frac{dx}{X} = \frac{dy}{Y};\ \text{and in (15)},\ \frac{dx}{X} = -\frac{dy}{Y}.$$

Let us in all suppose $x = \dfrac{\sin \omega}{\sin(\omega + a)}$;

then if $y = \dfrac{\sin\theta}{\sin(\theta - a)}$, $xy = 1$, when $\theta = \omega + a$

......, but $x + y = 2\cos a$, when $\theta = \omega + 2a$.

8. By equation (13) we can obtain $\Lambda(\sec a)$ and $\Lambda(\frac{1}{2}\sec a)$ from $\Lambda(\cos a)$ and $\Lambda(2\cos a)$. Observing that

$$L\cos^2 a + L\sec^2 a = 2\log^2\cos a,$$

we have
$$\left.\begin{aligned} \Lambda \sec a &= \tfrac{1}{4} L \sec^2 a + \tfrac{1}{2}\pi(\tfrac{1}{3}\pi - a) \\ \Lambda(\tfrac{1}{2}\sec a) &= \tfrac{1}{2}\log^2(\tfrac{1}{2}\sec a) + \tfrac{1}{12}\pi^2 - \tfrac{1}{2}a^2 \end{aligned}\right\} \ldots\ldots(17).$$

9. Farther, since we fulfil the relation $x^{-1} + y = 2\cos a$, by supposing

$$x = \frac{\sin\omega}{\sin(\omega + a)}, \quad y = \frac{\sin(\omega - a)}{\sin\omega},$$

it is evident that if Λx is known, we can by the repeated use of (15) find $\Lambda \dfrac{\sin\{\omega - (n+1)a\}}{\sin(\omega - na)}$. Or conversely, if Λy is known, we can deduce $\Lambda \dfrac{\sin(\omega + na)}{\sin\{\omega + (n+1)a\}}$.

For example: *first*, let m_n stand for $\dfrac{\cos na}{\cos(n-1)a}$; then $m_n^{-1} + m_{n+1} = 2\cos a$;

$$\therefore\ 2\Lambda m_{n+1} - 2\Lambda m_n = a^2 + L(m_n^{-1} m_{n+1}) - \log^2 m_n.$$

For n write $1, 2, 3 \ldots (n-1)$, and add the results, taking Λm_1 from (11);

$$\left.\begin{aligned} \therefore\ 2\Lambda \frac{\cos na}{\cos(n-1)a} &= \tfrac{1}{3}\pi^2 - \pi a + na^2 \\ &+ \tfrac{1}{2} L\cos^2 a + L(m_1^{-1} m_2) + L(m_2^{-1} m_3) + \ldots + L(m^{-1}_{n-1} m_n) \\ &- \log^2 m_1 - \log_2 m_2 - \&\text{c.} \ldots\ldots - \log^2 m_{n-1}: \end{aligned}\right\}$$

where for $\cos^2 a$ we may write $(m_0^{-1} m_1)$.

Similarly, if $m_n = \dfrac{\sin na}{\sin(n+1)a}$, $m_n^{-1} + m_{n-1} = 2\cos a$; and Λm_1 is known by (17); so that

$$2\Lambda \frac{\sin na}{\sin(n+1)a} = \tfrac{1}{6}\pi^2 - na^2 - L\frac{m_1}{m_2} - \ldots\ldots - L\frac{m_{n-1}}{m_n}$$
$$+ \log^2 m_1 + \log^2 m_2 + \ldots\ldots + \log^2 m_n.$$

10. A more general relation between Λx and Λy is obtainable by Mr. Fox Talbot's Method of Symmetrical Integrals.

Let $X = (m - x)\,v$, $Y = (m - y)\,v$; where m is constant, and x, y functions of v. Then x and y are the two roots of

$$x^2 - (2\cos a - v)\,x + (1 - mv) = 0\,;$$

consequently $x + y = 2\cos a - v$, $xy = 1 - mv$;

and, eliminating v,

$$(m - x)(m - y) = 1 - 2m\cos a + m^2 = M.$$

Now $\Lambda x + \Lambda y = \frac{1}{2}\int lx\,\{dlv + dl(m - x)\} + \frac{1}{2}\int ly\,\{dlv + dl(m - y)\}$

$$= \tfrac{1}{2}\int l(xy)\,dlv + \tfrac{1}{2}\int lx\,dl(m - x) + \tfrac{1}{2}\int ly\,dl(m - y).$$

The first integral

$$= \int l(1 - mv)\,dlv = L(1 - mv) = L(xy)\,;$$

the second

$$= lml(m - x) + L\frac{x}{m}\,;\ \text{the third} = lm\,l(m - y) + L\frac{y}{m}.$$

Observe that $lml(m - x) + lml(m - y) = lmlM = \text{const.}$;

$$\therefore\ 2\Lambda x + 2\Lambda y - \text{const.} = L(xy) + L\frac{x}{m} + L\frac{y}{m}.$$

Let $x = e$, when $y = 0$; then $e = 2\cos a - m^{-1}$; so that Λe and Λm will be known from one another by (15). Also

$$\left(1 - \frac{x}{m}\right)\left(1 - \frac{y}{m}\right) = 1 - \frac{e}{m} = 1 - 2e\cos a + e^2 = E.$$

Finally

$$2(\Lambda x + \Lambda y - \Lambda e) = L(xy) + L\frac{x}{m} + L\frac{y}{m} - L\frac{e}{m} - 2L0 \ldots (18).$$

By slightly varying the integration, we get

$$2(\lambda x + \lambda y - \lambda e)$$
$$= L(1 - xy) + L\left(1 - \frac{x}{m}\right) + L\left(1 - \frac{y}{m}\right) - L\left(1 - \frac{e}{m}\right) \ldots (18^*),$$

which may be convenient when m, or one of the variables, is negative.

By giving special values to m, such as make Λm and Λe known functions, the equations become available to us in many ways. But in order to understand our result, it will be well to transform it by means of ω and the function χ of Art. 1.

11. First observe that as m is arbitrary, it may be so settled (x being given), as to assign any required value to y. This amounts to making x and y independent variables, and m a function of them. And in fact,

$$m = \frac{1 - xy}{2 \cos a - (x + y)},$$

by which m may be eliminated, and e expressed in terms of x and y.

But, leaving m constant, put

$$x = \frac{\sin \omega}{\sin(\omega + a)}, \quad y = \frac{\sin \theta}{\sin(\theta + a)}, \quad e = \frac{\sin \eta}{\sin(\eta + a)};$$

Now $d\omega + d\theta = \dfrac{\sin a\, dx}{X} + \dfrac{\sin a\, dy}{Y} = \dfrac{\sin a}{v}\left(\dfrac{dx}{m - x} + \dfrac{dy}{m - y}\right)$

$$= \frac{-\sin a}{v} d \log\{(m - x).(m - y)\} = \frac{-\sin a}{v} d \log M = 0;$$

$\therefore\ \omega + \theta = \text{const.}$

When $\omega = 0$, $x = 0$; $\therefore\ y = e$, and $\theta = \eta$; or const. $= \eta$ and

$$\omega + \theta = \eta.$$

Again, $m^{-1} = 2 \cos a - e = \dfrac{\sin(\eta + 2a)}{\sin(\eta + a)}$. To save room, let $\psi(\omega)$ denote the function of ω to which x is equal;

$$\left.\begin{aligned} \therefore\ 2\{\chi\omega + \chi\theta - \chi(\omega + \theta)\} = L(\psi\omega.\psi\theta) - 2L0 \\ + L\frac{\psi\omega}{\psi(\omega + \theta + a)} + L\frac{\psi\theta}{\psi(\omega + \theta + a)} - L\frac{\psi(\omega + \theta)}{\psi(\omega + \theta + a)} \end{aligned}\right\}\ldots(19):$$

in which m and e or η have been eliminated, and ω, θ remain as independent variables.

Thus, if $\chi\omega$ and $\chi\theta$ are known for any particular values of ω and θ, $\chi(\omega + \theta)$ may be hence deduced. It is at once evident, that $\chi(\omega - \theta)$, $\chi(n\omega)$ and in fact $\chi\left(\dfrac{m}{n}\omega\right)$ may also be found, in finite terms, and by equations of the first degree. Yet this is rather a theoretical truth, than one of utility for calculation.

Since we already know Λx for the values $x = 1$, $x = 2 \cos a$, $x^{-1} = 2 \cos a$, $x = \cos a$, $x^{-1} = \cos a$; which correspond to $\omega = \frac{1}{2}(\pi - a)$, $\omega = \pi - 2a$, $\omega = a$, $\omega = \frac{1}{2}\pi - a$, $\omega = \pm\frac{1}{2}\pi$; we may start from any of these; and by addition or subtraction obtain a variety of results, and many more by combining division. If, however, we seek for $\chi(na)$ in this way, the result is far more complicated than in Art. 9. The cases which may chiefly deserve to be pointed out as attainable, are the following:

$\omega = \pi - a$	$\omega = \frac{1}{2}\pi + a$	$\omega = \frac{1}{2}(\pi - 3a)$.
$\omega = \frac{1}{2}a$	$\omega = \frac{1}{4}\pi \pm \frac{1}{2}a$	$\omega = \frac{1}{4}\pi - a$.
$\omega = \pm\frac{1}{4}\pi$	$\omega = \frac{1}{4}(\pi \pm a)$	

The process of bisection, transferred to (18), consists in supposing Λe known, and making $x = y = m(1 \mp \sqrt{E})$; then $2\Lambda x$ is found from Λe.

As equation (18) is virtually an integral of the equation

$$\frac{dx}{X} + \frac{dy}{Y} = 0,$$

and contains an additional arbitrary constant m; no more general result is attainable in this direction. If we had assumed $X = v(m - x)(n - y)$, $Y = v(m - y)(n - y)$, the integral of $d(\Lambda x + \Lambda y)$, hence arising, would be a mere combination of (18) with (15), and would tell us nothing new.

§ III.—*On changing* $\Lambda(x, a)$ *to* $\Lambda(y, \beta)$.

12. The properties hitherto attained have more show of utility than they make good, in regard to the general reduction of $\Lambda(x, a)$ to another calculable function. In that respect more advantage is derived from changing a in Λ simultaneously with x.

With a view to this, put now Y for $y^2 - 2y\cos\beta + 1$.

For a first integration, *assume* $x\cos a + y\cos\beta = 1$, and $a + \beta = \frac{1}{2}\pi$;

$$\therefore\ X = \sin^2 a\,(x^2 + y^2),\ Y = \sin^2\beta\,(x^2 + y^2);$$

$$dlX = dlY = dl\left(1 + \frac{x^2}{y^2}\right) + dl.y^2.$$

$$\text{Also}\quad \frac{dx}{\sin a} + \frac{dy}{\sin\beta} = 0;\quad \frac{\sin a\,dx}{X} + \frac{\sin\beta\,dy}{Y} = 0.$$

$$\text{Now}\ \Lambda(x, a) - \Lambda(y, \beta) = \tfrac{1}{2}\int(lx - ly)\,dl(x^2 + y^2)$$

$$= \tfrac{1}{2}\int l\frac{x}{y}.dl\left(\frac{x^2}{y^2} + 1\right) + \int l\frac{x}{y}\,dly.$$

The former integral $= \frac{1}{4}L(-x^2y^{-2})$. In the latter, observe that $(xy^{-1}) = \tan a\,\{(y\cos\beta)^{-1} - 1\}$. Put $y\cos\beta = v^{-1}$, $dly = -dlv$,

$\therefore \int l(xy^{-1})\,dly = -\int\{l\tan a + l(v-1)\}\,dlv$

$$= l\tan a.l(y\cos\beta) - L(1 - v) + \text{const.}$$

To correct, make $x = 0$, $y = \sec\beta$; then, since $v - 1 = \dfrac{x\cos\alpha}{y\cos\beta}$,

$$\left.\begin{array}{l}\Lambda(x,\alpha) - \Lambda(y,\beta) + \Lambda(\sec\beta,\beta) \\ = \tfrac{1}{4}L\left(-\dfrac{x^2}{y^2}\right) - L\left(-\dfrac{x\cos\alpha}{y\cos\beta}\right) + l\tan\alpha ly\cos\beta + \tfrac{3}{4}L0\end{array}\right\} \ldots (20).$$

To avoid the negatives under L, we may use the equation

$$L(-z) = lzl(1+z) - L(1+z) + L0;$$

reducing by which, we get for the right-hand member

$$\tfrac{1}{2}l(xy^{-1}).lX - \tfrac{1}{4}L(Xy^{-2}\cos^{-2}\beta) + L(y\cos\beta)^{-1}\ldots.(20^*).$$

This may be named the *Complementary Equation.*† If in it we make $\beta = \frac{1}{3}\pi$, $\Lambda(y,\beta)$ is a known function of y. Hence we can by it determine $\Lambda(x, \frac{1}{6}\pi)$ as a known function of x.

If in the last, $x = \dfrac{\sin\omega}{\sin(\omega+\alpha)}$, $y = \dfrac{\sin\theta}{\sin(\theta+\beta)}$, then from $\dfrac{\sin\alpha dx}{X} + \dfrac{\sin\alpha dy}{Y} = 0$, we get $d\omega + d\theta = 0$, or $\omega + \theta = \text{const.}$ Let $y = 0$, $x\cos\alpha = 1$; $\sin\omega\cos\alpha = \sin(\omega+\alpha)$, or $\omega = \frac{1}{2}\pi$, and $\theta = 0$;

$$\therefore\ \omega + \theta = \tfrac{1}{2}\pi;\quad \alpha + \beta = \tfrac{1}{2}\pi;\quad \sin(\omega+\alpha) = \sin(\theta+\beta).$$

13. For a second integration, *assume*

$$x = 2y\cos\beta - 1,\quad \alpha = 2\beta;$$

$$\therefore\ X = 4Y\cos^2\beta,\quad dlX = dlY,\quad dx = 2dy\cos\beta;$$

$$\frac{\sin\alpha\, dx}{X} = \frac{2\sin\alpha\, dy\cos\beta}{4Y\cos^2\beta} = \frac{\sin\beta dy}{Y},$$

which gives $d\omega = d\theta$. When $\omega = 0$, $x = 0$, $y^{-1} = 2\cos\beta$,

$\tan\theta = 2y\sin\beta$, or $\theta = \beta$; $\therefore\ \omega = \theta - \beta$; or $\omega + \alpha = \theta + \beta$.

So much for the algebraic relations of the variables.

$$\text{Now}\ \Lambda(x,\alpha) - 2\Lambda(y,\beta) = \tfrac{1}{2}\!\int\!\log(xy^{-2})\,dlY.$$

Let $y^{-1} = z$, $Z = z^2 - 2z\cos\beta + 1$; whence $Y = Zy^2$,

$$xy^{-2} = 1 - Z = (2y\cos\beta - 1)\,y^{-2};$$

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† In many formulas, it would appear that if our tables of Lx gave $\log x$ rather than x for the *argument*, this would be more convenient in the application. This suggests that *the same* table might give x, hyp log x and Lx.

$$\therefore \tfrac{1}{2}\int l(xy^{-2})\, dlY = \tfrac{1}{2}\int l(1-Z)\, dlZ + \int \log \frac{2y\cos\beta - 1}{y^2}\, dly$$

$$= \tfrac{1}{2}L(1-Z) + L(1-2y\cos\beta) - \log^2 y.$$

Hence

$$\Lambda(x, a) - 2\Lambda(y, \beta) = \tfrac{1}{2}L(xy^{-2}) + L(-x) - \log^2 y + C. \ldots (21)^*.$$

To find C, let $x = 0$, $y = \frac{1}{2}\sec\beta$; and after a few reductions, $C = \frac{1}{4}a^2 + \frac{1}{12}\pi^2$. The same result is found by making $x = -1$, $y = 0$; but the infinites under L and *log* are a little more troublesome.

This may be called the *Equation of Bisection,* since $a = 2\beta$, and since $\Lambda(x, a) - 2\Lambda(y, \beta)$ is expressed in known functions.

It follows that if $\Lambda(x, a)$ is a known function of x for some one value of a, so is $\Lambda(x, \frac{1}{2}a)$. For we have only to make $x' = 2x\cos\frac{1}{2}a - 1$, and determine $\Lambda(x, \frac{1}{2}a)$ from $\Lambda(x', a)$ by equation (21).

Hence $\Lambda\left(x, \dfrac{\pi}{2^n}\right)$ and $\Lambda\left(x, \dfrac{\pi}{3.2^n}\right)$ can be obtained in finite terms as known functions of x, since we know $\Lambda(x, \frac{1}{2}\pi)$ and $\Lambda(x, \frac{1}{3}\pi)$.

14. By a repeated use of the equation of bisection, it is evident that $\Lambda(x, a)$ is reducible to $\Lambda(x_n, 2^{-n}a)$, which, when $n = \infty$, is $\Lambda(x_n, 0)$ a known function. It may be worth while to enter into a few details concerning this.

Let a_n represent $2^{-n}a$, and from x suppose $x_1, x_2 \ldots$ to be derived by the law

$$2x_1\cos a_1 = 1 + x;\ 2x_2\cos a_2 = 1 + x_1;\ \&c.\ \&c.\ldots$$

It is easy to compute these by the intervention of ω. For we had

$$\omega + a = \theta + \beta \text{ or } = \omega_1 + a_1 = \omega_2 + a_2 = \omega_3 + a_3 = \&c.,$$

$$\text{whence } \omega_n = \omega + a - a_n.$$

Thus $x_n = \dfrac{\sin(\omega + a - 2^{-n}a)}{\sin(\omega + a)}$, which, when $n = \infty$, converges to 1, and nearly $= 1 - 2^{-n}a\cot(\omega + a)$. (We must entirely except the case of $\omega + a = 0$ or $= \pi$, which gives $x = \infty$.) Hence $2^n.\Lambda(x_n\, a_n) = 2^n.\Lambda(x_n\, 0) = 2^n.\{Lx_n + \frac{1}{6}\pi^2\} = 2^n.(x_n - 1) + 2^n.\frac{1}{6}\pi^2 = -a\cot(\omega + a) + 2^n.\frac{1}{6}\pi^2$. Apply equation (21) n times: multiply the results by $2^0, 2^1, 2^2 \ldots 2^{n-1}$, and add all together. Substitute for $2^n.\Lambda(x_n\, a_n)$ as above, and be careful to note that $2^{-2} + 2^{-3} + \ldots + 2^{-\infty} = \frac{1}{2}$,

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* It is easy to combine equations (20), (21) with (13).

and $(2^0 + 2^1 + 2^2 + \ldots + 2^{n-1}) \frac{1}{12}\pi^2 + 2^n . \frac{1}{6}\pi^2$

$$= \tfrac{1}{6}\pi^2 + (2^0 + 2^1 + \ldots + 2^{n-1}) \tfrac{1}{4}\pi^2 :$$

after which, making $n = \infty$, we get

$$\left.\begin{aligned}\Lambda(x, a) = \tfrac{1}{2}a^2 - a \cot(\omega + a) + \tfrac{1}{6}\pi^2 & \\ + 2^0 \{\tfrac{1}{4}\pi^2 + L(-x) - \log^2 x_1 + \tfrac{1}{2}L(x\, x_1^{-2})\} & \\ + 2^1 \{\tfrac{1}{4}\pi^2 + L(-x_1) - \log^2 x_2 + \tfrac{1}{2}L(x_1 x_2^{-2})\} & \\ + 2^2 \{\tfrac{1}{4}\pi^2 + L(-x_2) - \log^2 x_3 + \tfrac{1}{2}L(x_2 x_3^{-2})\} & \\ + \&c. \&c. \ldots\ldots & \end{aligned}\right\} \ldots(22).$$

This always converges, yet not rapidly. When x_n is approaching its limit 1, we may approximately determine the remnant of the series, by the formulas

$L(1 - h) = - h - \frac{1}{4}h^2 - \frac{1}{9}h^3$; $\frac{1}{4}\pi^2 + L(-1 + h) = \frac{1}{4}h^2 + \frac{1}{6}h^3$; when h is very small.
$\log^2(1 - h) = h^2 + h^3$. Also $h_n = a_n \{(1 - \frac{1}{8}a_n^2)\cot(\omega + a) + \frac{1}{2}a_n\}$; and $h_{n+1} = (\frac{1}{2}h_n - \frac{1}{8}a_n^2)(1 + \frac{1}{8}a_n^2)$, very nearly.

But the great defect of the method is, that even if we start with x nearly $= 1$, we still do not any the more rapidly reach the limit $x_n = 1$: hence the series has no practical interest, unless indeed at once both x is very near to 1, and $\cos a$ between x and 1, a case which is the most troublesome of all in the method of Art. 16.

15. The equation of bisection would farther enable us to increase the number of functions $x = Fa$, which give $\Lambda(x, a)$ as a known function of a. For let $x = Fa$ be any one function, for which $\Lambda(x, a)$ is known; put $2x_1 \cos \frac{1}{2}a = 1 + x$, or put $x' = 2x \cos a - 1$; and $x_1\, x'$ are new functions of a, for which $\Lambda(x_1, \frac{1}{2}a)$ and $\Lambda(x', 2a)$ are known.

If we could integrate so as to obtain $\Lambda(x, a) + m\Lambda(y, \beta)$ in known functions, when $d\omega \propto d\theta$, by means of some *general* relations uniting a, β, m; it would more than anything else perfect what is wanting in this theory.

§ IV.—*To calculate* $\Lambda(x, a)$ *in any case.*

16. We have now the means of reducing $\Lambda(x, a)$ in all cases to another function $\Lambda(x', a')$ in which x' shall be less than $\frac{1}{2}$; which will enable us to apply equation (12).

Avoiding details, it will suffice here to shew the possibility of the transformation.

First, when $\alpha > 60°$; if x is > 1, we may reduce Λ to the case of $x < 1$ by equation (13). If then the new x is between $\frac{1}{2}$ and 1, put $x' = -x$, $\alpha' = \pi - \alpha = 2\beta$, $x' = 2y \cos \beta - 1$; $\Lambda(x, \alpha) = \Lambda(x', \alpha')$. Apply the equation of bisection to reduce $\Lambda(x, \alpha)$ to $\Lambda(y, \beta)$. Now as α' is $< 120°$, β is $< 60°$, $2 \cos \beta > 1$, $2y \cos \beta > y$; $\therefore 1 + x'$ or $1 - x > y$, or $y < \frac{1}{2}$.

Next, when α is $< 30°$, put $\alpha + \beta = \frac{1}{2}\pi$, and use the Complementary Equation. Since β is $> 60°$, this case is reduced to the former.

Thirdly, when α is between 60° and 30°. Here x is by hypothesis between 1 and $\frac{1}{2}$, and $2 \cos \alpha$ between $\sqrt{3}$ and 1; so that $2x \cos \alpha$ is between $\sqrt{3}$ and $\frac{1}{2}$.—We separate the case of $x > \cos \alpha$; in which we can proceed exactly as when α was $> 60°$. For since $2y \sin \frac{1}{2}\alpha = 1 - x$, which is $< 1 - \cos \alpha$ or than $2 \sin^2 \frac{1}{2}\alpha$, $\therefore y$ is $< \sin \frac{1}{2}\alpha < \frac{1}{2}$.—When x is *not* $> \cos \alpha$, $2x \cos \alpha$ does not exceed $2 \cos^2 \alpha$ or $\frac{3}{2}$; so that its limits are $\frac{3}{2}$ and $\frac{1}{2}$. Put $y = 2x \cos \alpha - 1$, and y is between $+\frac{1}{2}$ and $-\frac{1}{2}$. If then $2\alpha = \beta$, we can reduce by equation (21), only exchanging x with y, and α with β.

The simplicity of the coefficients in equation (12), which are known by common tables, would lead us to prefer that series when other things are equal. Yet if x is near to $\frac{1}{2}$, its convergence is not such as to give accuracy to many decimal places without great labour; and some of the following methods may become preferable.

§ V.—*To take advantage of* α *lying within certain limits.*

17. If α is extremely small, and x is $< \frac{1}{2}$; or if, x being near to 1, the product $2 \sin \frac{1}{2}\alpha . \left(\frac{x}{1-x}\right)$ is still very small.

$$\text{Put } b = 2 \sin \tfrac{1}{2}\alpha,\ z = \frac{x}{1-x},\ \text{or } x = \frac{z}{1+z};\ 1 - x = \frac{1}{1+z};$$

$$X = (1-x)^2 + b^2 x = (1-x)^2.\{1 + b^2 z.(1+z)\}$$

$$d \log x = d\{\log z - \log(1+z)\} = \frac{dz}{z(1+z)};$$

$$\therefore \lambda(x, \alpha) = \tfrac{1}{2}\textstyle\int_0 \log X\, d \log x = \int_0 \log(1-x)\, d \log x$$

$$+ \tfrac{1}{2}\textstyle\int_0 \log\{1 + b^2 z.(1+z)\} \dfrac{dz}{z(1+z)}$$

$$= L(1-x) + \tfrac{1}{2}P \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

$$\text{if } P = \textstyle\int_0 \{b^2 - \frac{1}{2}b^4 z.(1+z) + \frac{1}{3}b^6 z^2.(1+z)^2 - \&c.\ldots\} dz$$

$$= b^2 z - \tfrac{1}{2}b^4(\tfrac{1}{2}z^2 + \tfrac{1}{3}z^3) + \tfrac{1}{3}b^6(\tfrac{1}{3}z^3 + 2\,\tfrac{1}{4}z^4 + \tfrac{1}{5}z^5) - \&c.\ldots (23^*),$$

which converges rapidly, since bz is very small.

18. If, on the contrary, α is very near to π (which is always the more favourable case, x being supposed positive), let $x = \tan^2 \frac{1}{2}\omega$; then $\lambda(x, \alpha) = L(1 + x) - 2\Omega$,

$$\text{if } \Omega = -\tfrac{1}{2}\int_0 \log(1 - \cos^2 \tfrac{1}{2}\alpha \sin^2 \omega)\frac{d\omega}{\sin \omega}.$$

If we develop the logarithm, we readily see that Ω may take the form

$$A_0 - 2A_1 \cos \omega + 2A_3 \frac{\cos 3\omega}{3} - 2A_5 \frac{\cos 5\omega}{5} + \&c.$$

To find A_0, let $\omega = \frac{1}{2}\pi$, $\Omega = A_0$, $x = 1$, $\therefore \lambda(1, \alpha) = L2 - 2A_0$;

whence $2A_0 = \dfrac{\pi^2}{12} + \Lambda(1, \alpha) = \left(\dfrac{\pi - \alpha}{2}\right)^2$.—Let $\pi - \alpha = 4\beta$,

$$\therefore A_0 = 2\beta^2.$$

Next $\dfrac{d\Omega}{d\omega} = 2A_1 \sin \omega - 2A_3 \sin 3\omega + 2A_5 \sin 5\omega - \&c.,$

$$\text{also } \frac{d\Omega}{d\omega} = -\tfrac{1}{2}\log(1 - \sin^2 2\beta \, . \sin^2 \omega)\frac{1}{\sin \omega}.$$

Put $b = \tan \beta$, $\sin 2\beta = \dfrac{2b}{1 + b^2}$; and the value of $\sin \omega \, . \dfrac{d\Omega}{d\omega}$

$$\text{is } \log(1 + b^2) - \tfrac{1}{2}\log(1 + 2b^2 \cos 2\omega + b^4),$$

or $\log(1 + b^2) - b^2 \cos 2\omega + \frac{1}{2}b^4 \cos 4\omega - \frac{1}{3}b^6 \cos 6\omega + \&c.,$
which is to be made equal to

$$2 \sin \omega \{A_1 \sin \omega - A_3 \sin 3\omega + A_5 \sin \omega - \&c.\},$$

or $A_1(1 - \cos 2\omega) - A_3(\cos 2\omega - \cos 4\omega) + A_5(\cos 4\omega - \cos 6\omega) - \&c.$
Hence we get $\quad A_1 = \log(1 + b^2)$,

$$A_1 + A_3 = \frac{b^2}{1}; \text{ and generally } A_{2n-1} + A_{2n+1} = \frac{b^{2n}}{n}.$$

In the First Part of these investigations we have used $\phi_n x$ to denote $\int_0 \tan^{n-1} x \, dx$; which yields $\phi_1 x = x$, $\phi_2 x = \frac{1}{2}\log(1 + \tan^2 x)$ or $\log \sec x$; and $\phi_n x + \phi_{n+2} x = \dfrac{\tan^n x}{n}$.

Thus $A_1 = 2\phi_2\beta$; $A_3 = 2\phi_4\beta$; $A_5 = 2\phi_6\beta$; &c. . . .

$$\left.\begin{aligned}\text{and } \lambda(x, \alpha) = L(1 + x) - 4\beta^2 + 8\phi_2\beta \, . \frac{\cos \omega}{1} - 8\phi_4\beta \, . \frac{\cos 3\omega}{3} \\ + 8\phi_6\beta \, . \frac{\cos 5\omega}{5} - 8\phi_8\beta \, . \frac{\cos 7\omega}{7} \\ + \&c. \ldots \&c. \ldots \end{aligned}\right\} \ldots (24),$$

which converges best when β is least, or α nearest to π.

19. To find Λ and λ, when a is near to $\frac{1}{2}\pi$.

Put $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$;

$$\therefore \lambda(x, a) = \tfrac{1}{4}L(1+x^2) - \Omega, \text{ if } \Omega = \tfrac{1}{2}\int \log(1 - \cos a \cos\omega)\frac{d\omega}{\cos\omega}.$$

Assume $\Omega = C - C_0\omega - 2C_1\sin\omega - 2C_2\dfrac{\sin 2\omega}{2} - 2C_3\dfrac{\sin 3\omega}{3} - \&c.$

To determine C, put $\omega = 0$, $x = 1$, $\Omega = C$, $\lambda(1, a) = \frac{1}{4}L2 - C$,

or $C = \frac{1}{4}L2 + \Lambda(1, a) = \frac{1}{16}3\pi^2 - \frac{1}{2}\pi a + \frac{1}{4}a^2$.

To determine C_0,

$$\text{we have } -\frac{d\Omega}{d\omega} = C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c.\ldots$$

Multiply by $d\omega$, and integrate from $\omega = 0$ to $\omega = \pi$, observing that $\int_0^\pi \cos n\omega\, d\omega = 0$, for all integer values of n; also $\int_0^\pi C_0 d\omega = \pi C_0$.

Then $\pi C_0 = \displaystyle\int_0^\pi -\frac{d\Omega}{d\omega}d\omega = \int_1^{-1} -\frac{d\Omega}{dx}dx = \Omega$ (from $x = -1$ to $x = 1$) $= \{\frac{1}{4}L2 - \lambda(1, a)\} - \{\frac{1}{4}L2 - \lambda(-1, a)\} = \Lambda(1, a) - \Lambda(1, \pi - a) = (\frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{4}a^2) - (\frac{1}{4}a^2 - \frac{1}{12}\pi^2) = \frac{1}{4}\pi^2 - \frac{1}{2}\pi a$.

Whence $C_0 = \frac{1}{2}(\frac{1}{2}\pi - a)$. Call this γ. $\therefore C = \frac{1}{2}\pi\gamma + \gamma^2$.

Farther, put $c = \tan\gamma$, $\cos a = \sin 2\gamma = \dfrac{2c}{1+c^2}$;

$$\cos\omega.\frac{d\Omega}{d\omega} = \tfrac{1}{2}\log\left(1 - \frac{2c\cos\omega}{1+c^2}\right)$$

$$= -\tfrac{1}{2}\log(1+c^2) - c\cos\omega - \tfrac{1}{2}c^2\cos 2\omega - \tfrac{1}{3}c^3\cos 3\omega - \&c.\ldots$$

But $-\cos\omega\dfrac{d\Omega}{d\omega} = \cos\omega\{C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c.\ldots\}$

$= C_1 + (C_0 + C_2)\cos\omega + (C_1 + C_3)\cos 2\omega + (C_2 + C_4)\cos 3\omega + \&c.\ldots$

$\therefore C_1 = \frac{1}{2}\log(1+c^2) = \phi_2\gamma$; $C_2 = c - C_0 = \tan\gamma - \gamma = \phi_3\gamma$;

$C_3 = \frac{1}{2}c^2 - C_1 = \phi_4\gamma$; $C_4 = \frac{1}{3}c^3 - C_2 = \phi_5\gamma$; &c. . . .

$$\left.\begin{aligned}\text{Whence } \lambda(x, a) &= \tfrac{1}{4}L(1+x^2) - (\tfrac{1}{2}\pi\gamma + \gamma^2) + \gamma\omega \\ &+ 2\phi_2\gamma.\frac{\sin\omega}{1} + 2\phi_3\gamma.\frac{\sin 2\omega}{2} + 2\phi_4\gamma.\frac{\sin 3\omega}{3} + \&c.\end{aligned}\right\}\ldots(25),$$

where $\gamma = \frac{1}{4}\pi - \frac{1}{2}a$, $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$.

The convergence is rapid when a is very near to $\frac{1}{2}\pi$.

In equation (24), put $\omega = 0$, $\Omega = 0$;

$$\therefore \tfrac{1}{2}\beta^2 = \phi_2\beta - \tfrac{1}{3}\phi_4\beta + \tfrac{1}{5}\phi_6\beta - \&c.$$

In the value of $\dfrac{d\Omega}{d\omega}$ corresponding, make $\omega = \frac{1}{2}\pi$;

$$\therefore -\tfrac{1}{4}\log\cos 2\beta = \phi_2\beta + \phi_4\beta + \phi_6\beta + \&c.\ldots$$

In equation (25), if we change γ into $-\gamma$, $\phi_{2n}\gamma$ remains unchanged, and $\phi_{2n-1}\gamma$ changes sign. By adding the two results thus obtained, we might easily reproduce equation (24).

Put $\omega = \pi$ in the value of $\frac{d\Omega}{d\theta}$ corresponding to (25);

$$\therefore \tfrac{1}{2}\log(1 + \sin 2\gamma) = \phi_1\gamma - 2\phi_2\gamma + 2\phi_3\gamma - 2\phi_4\gamma + \&c.,$$

$$\text{so } \tfrac{1}{2}\log(1 - \sin 2\gamma) = -\phi_1\gamma - 2\phi_2\gamma - 2\phi_3\gamma - 2\phi_4\gamma - \&c.;$$

which gives not only

$$-\tfrac{1}{4}\log\cos 2\gamma = \phi_2\gamma + \phi_4\gamma + \phi_6\gamma + \&c.,$$

but also $\tfrac{1}{2}\log\tan(\tfrac{1}{4}\pi + \gamma) = \phi_1\gamma + 2\phi_3\gamma + 2\phi_5\gamma + \&c.\ldots.$

These are mere properties of the functions $\phi_1, \phi_2, \phi_3 \ldots$ and can in several ways be verified.

The series (24), (25) cannot be practically used with advantage, unless we have tables of $\phi_n\alpha$; but these might be computed with so much ease, within the limits $\alpha = 0$, $\alpha = 45°$, that this is apparently the best method of adding completeness to this branch of the calculus. The following section will shew that the use of ϕ_n is not confined to the particular cases contemplated in equations (24), (25).

§ VI.—*To find* Λ, *when* x *is near to* 1.

20. We shall suppose α to be $< 90°$, and deal with

$$\Lambda(x, \pi - \alpha) \text{ and } \Lambda(x, \alpha) \text{ separately.}$$

Put $\cos\alpha = \frac{1 - m^2}{1 + m^2}$, or $m = \tan\tfrac{1}{2}\alpha$; $X' = 1 + 2x\cos\alpha + x^2$;

$$\therefore (1 + m^2)X' = (1 + x)^2 + m^2(1 - x)^2. \quad \text{Let } y = \frac{1 - x}{1 + x}.$$

Then $\Lambda(x, \pi - \alpha) = \tfrac{1}{2}\log x \log X' - L(1 + x) + R,$

$$\text{if } R = \int \log\frac{1 + m^2y^2}{1 + m^2} \cdot \frac{dy}{1 - y^2}.$$

$$\text{Assume } -\frac{dR}{dy} = M_0 - M_2y^2 + M_4y^4 - \&c.\ldots;$$

$$\therefore \log(1 + m^2) - \frac{m^2y^2}{1} + \frac{m^4y^4}{2} - \frac{m^6y^6}{3} + \&c.\ldots$$

$$= (1 - y^2)\{M_0 - M_2y^2 + M_4y^4 - \&c.\ldots\},$$

which gives

$M_0 = \log(1 + m^2) = 2\phi_2\tfrac{1}{2}\alpha$; $M_2 = 2\phi_4\tfrac{1}{2}\alpha$; $M_4 = 2\phi_6\tfrac{1}{2}\alpha$; &c....

To find the constant after integrating $\frac{dR}{dy}$, make $x = 1$,

$$y = 0\,;\ \therefore R = \text{const.} = \Lambda(1, \pi - a) + L2 = \tfrac{1}{4}a^2.$$

Hence
$$\left.\begin{aligned}\Lambda(x, \pi - a) = \tfrac{1}{2}\log x \log X' - L(1+x) + \tfrac{1}{4}a^2 \\ - 2\phi_2 \tfrac{1}{2}a \,.\, \frac{y}{1} + 2\phi_4 \tfrac{1}{2}a \,.\, \frac{y^3}{3} - 2\phi_6 \tfrac{1}{2}a \,.\, \frac{y^5}{5} + \&c.\ldots\end{aligned}\right\} \ldots (26).$$

When $x = 0$, $y = 1$,

$$\tfrac{1}{2} \,.\, \tfrac{1}{4}a^2 = \phi_2 \tfrac{1}{2}a - \tfrac{1}{3}\phi_4 \tfrac{1}{2}a + \tfrac{1}{5}\phi_6 \tfrac{1}{2}a - \&c.\ldots.$$

which serves to verify the conclusion.

21. Next, observe that $(1 + m^2) \,.\, X = (1 + x)^2 \,.\, (m^2 + y^2)$, so that

$$\Lambda(x, a) = \left(\frac{\pi - a}{2}\right)^2 + \log x \log(1 + x) - L(1 + x) - S \ldots (27),$$

$$\text{if}\quad S = \tfrac{1}{2}\int_0 \log\frac{1+y}{1-y} \,.\, d\log(m^2 + y^2)\,;$$

where the arbitrary constant is found, as before, by making $x = 1$.

Developing $\log\frac{1+y}{1-y}$,

$$\tfrac{1}{2}S = \int_0 (y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \&c.\ldots)\frac{y\,dy}{m^2 + y^2}$$

$$= \int_0 \frac{y^2\,dy}{m^2 + y^2} + \tfrac{1}{3}\int_0 \frac{y^4\,dy}{m^2 + y^2} + \tfrac{1}{5}\int_0 \frac{y^6\,dy}{m^2 + y^2} + \&c.\ldots.$$

Add $\tfrac{1}{2}a \tan^{-1}\frac{y}{m} = \int_0 \frac{m^2\,dy}{m^2 + y^2} - \tfrac{1}{3}\int_0 \frac{m^4\,dy}{m^2 + y^2} + \tfrac{1}{5}\int_0 \frac{m^6\,dy}{m^2 + y^2} - \&c.\ldots$

and the sum is $\int_0 dy - \tfrac{1}{3}\int_0 (m^2 - y^2)\,dy + \tfrac{1}{5}\int_0 (m^4 - m^2y^2 + y^4)\,dy - \&c.$

$$= \int_0 (M_1 + M_3 y^2 + M_5 y^4 + \ldots)\,dy,$$

$$\begin{aligned}\text{if } M_1 &= 1 - \tfrac{1}{3}m^2 + \tfrac{1}{5}m^4 - \tfrac{1}{7}m^6 + \&c.\ldots. = m^{-1} \,.\, \tfrac{1}{2}a, \\ M_3 &= \tfrac{1}{3} - \tfrac{1}{5}m^2 + \tfrac{1}{7}m^4 - \tfrac{1}{9}m^6 + \&c.\ \ldots. = m^{-3} \,.\, \phi_3 \tfrac{1}{2}a, \\ M_5 &= \tfrac{1}{5} - \tfrac{1}{7}m^2 + \tfrac{1}{9}m^4 - \tfrac{1}{11}m^6 + \&c.\ldots. = m^{-5} \,.\, \phi_5 \tfrac{1}{2}a,\end{aligned}$$

and so on.

Write $z = ym^{-1} = \frac{1}{m} \,.\, \frac{1-x}{1+x}$; then we finally get

$$S = -a\tan^{-1}z + 2\left\{\phi_1 \tfrac{1}{2}a \,.\, \frac{z}{1} + \phi_3 \tfrac{1}{2}a \,.\, \tfrac{1}{3}z^3 + \phi_5 \tfrac{1}{2}a \,.\, \tfrac{1}{5}z^5 + \&c.\ldots\right\} \ldots (27^*).$$

The multipliers m^{-1}, m^{-3}, m^{-5}... here injure the convergence, as compared with that of (26). Yet in the worst case, $m = 0$, the coefficients M_1, M_3, M_5....become 1, $\frac{1}{3}$, $\frac{1}{5}$....so that the series always converges faster than

$$y + 3^{-2}y^3 + 5^{-2}y^5 + \ldots \&c.\ldots$$

and when x is $>\frac{1}{2}$, y^2 is $<\frac{1}{9}$; which is a far better convergence than we ordinarily get from equation (12).

22. We may increase the convergence (for the latter case only) by representing the given function as $\Lambda(x^2, 2a)$, and using the formula

$$\tfrac{1}{2}\Lambda(x^2, 2a) = \Lambda(x, \pi - a) + \Lambda(x, a),$$

taking $\Lambda(x, \pi - a)$ from (26) and $\Lambda(x, a)$ from (27). Thus if we wish to estimate $\Lambda(h, \mu)$, where h and μ are given, put $x^2 = h$, $2a = \mu$; then $y = \dfrac{1-\sqrt{h}}{1+\sqrt{h}}$, which is smaller than if we had made $x = h$, or $y = \dfrac{1-h}{1+h}$. In fact, this will enable us to restrict the use of equation (12) to the case of $x < \frac{1}{4}$; for if the variable is $> \frac{1}{4}$, call it x^2; $\therefore y^2$ is $< \frac{1}{9}$, and we find Λ by combining (26) and (27).

Supposing tables of ϕ_n to have been formed, it would perhaps be worth while, for the sake of the method just suggested, to add to them the values of f_n; where

$f_1 a = \cot a\, \phi_1 a + \phi_2 a$	$3f_3 a = \cot^3 a\, \phi_3 a - \phi_4 a$
$5f_5 a = \cot^5 a . \phi_5 a + \phi_6 a$	$7f_7 a = \cot^7 a\, \phi_7 a - \phi_8 a$
&c....	&c....

whence we obtain

$$\left.\begin{aligned}&\tfrac{1}{2}\Lambda(x^2, 4a) = a^2 + (\tfrac{1}{2}\pi - a)^2\\ &+ lxl(1+x) + \tfrac{1}{2}lxl(1 + 2x\cos 2a + x^2) - 2L(1+x)\\ &+ 2a.\tan^{-1}(y\cot a) - 2\{yf_1a + y^3f_3a + y^5f_5a + \ldots\}\end{aligned}\right\}\ldots(28).$$

This is more compact to the eye: yet we here lose the advantage of regularity in the decrease of the coefficients.

§ VII.—*To find Λ when x is near to $\cos a$.*

23. When x is near to $2\cos a$, we may reduce $\Lambda(x, a)$ by means of equation (14); but no such property has occurred with reference to $\cos a$.

Let $y = 1 - \dfrac{x}{\cos a}$; $X = (y\cos a)^2 + \sin^2 a$;

$$dlX = dl(y^2 + \tan^2 a).$$

Put $y\cos a = v$, $x^2 = (\cos a - v)^2 = (\cos^2 a - v^2) \div \dfrac{\cos a + v}{\cos a - v}$;

$$\therefore\ 2lx = l(\cos^2 a - v^2) - l\left(\frac{1+y}{1-y}\right).$$

Say $T = \frac{1}{4}\int l(\cos^2 a - v^2)\, dl\,(v^2 + \sin^2 a)$;

$$\left.\begin{aligned} U = \tfrac{1}{4}\int_0 l\left(\frac{1+y}{1-y}\right) dl\,(y^2 + \tan^2 a); \end{aligned}\right\} \therefore\ \Lambda(x, a) = c + T - U.$$

Let $\cos^2 a - v^2 = V$, $v^2 + \sin^2 a = 1 - V$, $T = \frac{1}{4}\int lV\, dl\,(1 - V)$ $= \frac{1}{4}L(V)$. To find c, let $y = 0$, $\therefore\ \Lambda(\cos a, a) = c + \frac{1}{4}L\cos^2 a$, or $c = \frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{2}a^2$. Observe also that $V = 1 - X$.

To find U, we have only to compare it with $\frac{1}{2}S$ of Art. 21, and write $\tan a$ for m; that is, a for $\frac{1}{2}a$. Hence if $u = y\cot a$ $= (\cos a - x)\operatorname{cosec} a$,

$$\left.\begin{aligned} \Lambda(x, a) &= (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{2}a^2) + \tfrac{1}{4}L(2x\cos a - x^2) \\ &\quad + a\tan^{-1} u + \frac{u}{1}\phi_1 a + \frac{u^3}{3}\phi_3 a + \frac{u^5}{5}\phi_5 a + \&c\ldots \end{aligned}\right\}(29).$$

§ VIII.—*Geometrical idea of the function* $y = \Lambda(x, a)$.

24. When a is $< 90°$, $\frac{dy}{dx}$ or $\frac{\log x\,.\,(x - \cos a)}{X}$ is positive from $x = 0$ to $x = \cos a$, and then negative until $x = 1$; after which it is perpetually positive. Thus Λx increases up to $\Lambda\cos a$, which is a maximum, and decreases down to $\Lambda 1$, which is (geometrically) a minimum. But it is not certainly a numerical minimum, if it has become negative.

Since $\Lambda 1 = \left(\frac{\pi - a}{2}\right)^2 - \frac{\pi^2}{12}$, this cannot be negative, unless $(\pi - a)^2 < \frac{1}{3}\pi^2$, or $a > (1 - 3^{-\frac{1}{2}})\pi$, which brings a near to the limit $\frac{1}{2}\pi$. If a is $< (1 - 3^{-\frac{1}{2}})\pi$, Λx never becomes negative; and $\Lambda 1$ is a numerical minimum.

When $x = 0$, $\frac{dy}{dx} = -\log x\,.\cos a = +\infty$ when a is $< 90°$, or is $-\infty$ when a is $> 90°$. When $x = \cos a$, or $x = 1$, $\frac{dy}{dx} = 0$. Again, the curve has an infinite branch corresponding to $x = \infty$, which gives $\Lambda x = 2\Lambda 1 + \frac{1}{2}\log^2 x$.

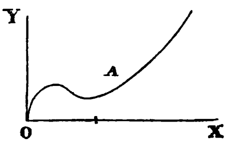
When a is $< 90°$, but so little less as to make $\Lambda 1$ negative, there are *two* values of x (one on each side of $x = 1$), such as to make $\Lambda x = 0$; besides the value $x = 0$.

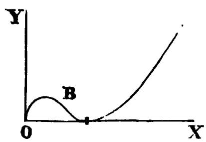
When a is $> 90°$, $\frac{dy}{dx}$ is negative from $x = 0$ to $x = 1$, after which it is always positive; and, as before, Λx is positive infinity when $x = \infty$. There is then *one* value x that makes

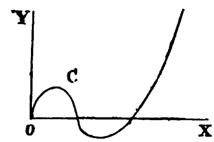
$\Lambda x = 0$, besides $x = 0$. This value of x is > 1, since $\Lambda 1$ is now essentially negative.

When $\alpha = \frac{1}{2}\pi$ exactly, $\frac{dy}{dx} = \frac{x \log x}{1 + x^2}$; which vanishes with x. A curve is thus produced essentially different from, and intermediate to, the other two species. Up to $x = 1$, $\frac{dy}{dx}$ is negative; and afterwards positive.

This suffices to give a rough notion of the three species of curves with which we are here concerned; viz. *first*, when α is $< 90°$, we have forms such as A, B, C.

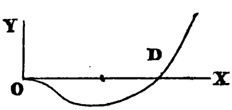


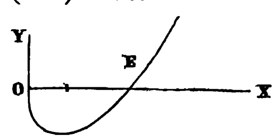




Of these, the form B exhibits $\Lambda 1$ exactly $= 0$; which gives $\alpha = \pi(1 - 3^{-\frac{1}{2}})$. In A, α is greater; and in C, α is less than this limiting value.

Secondly, when $\alpha = 90°$, the form appears to be such as D. This is in fact the curve $4y = L(-x^2) - L0$.





But *Thirdly*, when α is $> 90°$, we have the form E, which appears to be much simpler than the others.

ON A PROBLEM IN COMBINATIONS.*

By the Rev. Thomas P. Kirkman, A.B., Rector of Croft with Southworth, Lancashire.

If $Q_{x,y,z}$ denote the greatest number of combinations of y together, that can be made with x symbols, so that no combination of z together shall be twice employed; $Q_{x,y,1}$ is the greatest integer in $\frac{x}{y}$, and $Q_{x,y,y}$ is $\frac{x^{y|-1}}{y^{y|-1}}$, or the number of combinations of y together that can be made with x things. Thus division of integers, and this simple problem of com-

* Read before the Literary and Philosophical Society of Manchester, December 15, 1846.

MATHEMATICAL NOTES.

I. *Note relative to Mr. Newman's paper on Logarithmic Integrals of the Second Order.*

After these pages had passed through the press, Mr. Cayley mentioned to the Editor that nearly the same subject had been treated in *Crelle's Journal* (Vol. xxx. 1840), by Professor Kummer. After a rapid perusal, I can only add that this is certainly true, and that *many* of the properties above investigated have been discovered, and some others besides. Whether some of mine are not wholly new, I am unable to assert positively, by reason of the great difference of notation; nevertheless I believe that several of my equations concerning Λ and ל are not contained in Professor Kummer's investigations.

He states, that the integral $\int F_1 x . \log F_2 x . dx$ was treated by *Hill* in the *Journal der Mathematik*, Band III., and the integral $\int \log (1 + 2x \cos a + x^2)\, x^{-1} dx$, in a separate Latin treatise, by the same, in 1830. Kummer has enlarged on *Hill*, whose labours he regrets are so little known. It is curious that neither Kummer nor Hill seem to have known of *Spence's* integral, while virtually treating of the same under the form $\int (1 + x)^{-1} \log (\pm x). dx$. It appears moreover from Kummer (p. 220), that *Clausen* has actually tabulated my integral ל in p. 298 of *Crelle's Journal*, Vol. VIII., under the form $-\int_0 \log (\pm 2 \sin \frac{1}{2} a)\, da$.

Professor Kummer conceives of the general integral under the form $\int F_1 x \int F_2 x dx . dx$; and he has also extended his views to the third, fourth, fifth, &c. orders of rational integrals (for this appears to be the more appropriate title), and has exhibited in them integrals which are analogous to those of the second order.

F. W. NEWMAN.

May 8*th*, 1847.

II. *On the Caustic by Reflection at a Circle.*

[*To the Editor.*]

A paper by Mr. Cayley, under the above title, having been published in the last number of your *Journal*, it appears to me that both M. de St. Laurent and Mr. Cayley have overlooked the admirably symmetrical solution of the problem given by Lagrange in the *Mem. de Turin*. Thinking that some of your correspondents may be interested in it, I beg to send you a translation.